

The Most General $4\mathcal{D} \mathcal{N} = 1$ Superconformal Blocks for Scalar Operators

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Abstract

We compute the most general superconformal blocks for scalar operators in $4\mathcal{D} \mathcal{N} = 1$ superconformal field theories. Specifically we employ the supershadow formalism to study the four-point correlator $\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle$, in which the four scalars Φ_i have arbitrary scaling dimensions and R-charges with the only constraint from R-symmetry invariance of the four-point function. The exchanged operators can have arbitrary R-charges. Our results extend previous studies on $4\mathcal{D} \mathcal{N} = 1$ superconformal blocks to the most general case, which are the essential ingredient for superconformal bootstrap, especially for bootstrapping mixed correlators of scalars with independent scaling dimensions and R-charges.

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I. INTRODUCTION

The conformal bootstrap program, which was initially proposed for two dimensional conformal field theories (CFTs) [1–3] has been found to be remarkably powerful to study CFTs in higher dimensional spacetime [4]. The crossing symmetry and unitarity condition can provide strong constraints on the operator scaling dimensions, coefficients in operator product expansion (OPE) and the central charges [5–25]. The most striking results are obtained in [13, 18], in which the classical 3D Ising model and $O(N)$ vector model are studied through bootstrapping the mixed correlators. It is shown that by imposing certain reasonable assumptions on the spectrum, the CFT data can be isolated in small islands. These results are expected to be generalized to the supersymmetric theories, in which supersymmetry provides strong constraints on the quantum dynamics and leads to abundant conformal theories. The supersymmetric conformal bootstrap is especially important for 4D theories since most of the known 4D CFTs are of supersymmetric conformal field theories (SCFTs).

The critical ingredient utilized in conformal bootstrap is the convexity of conformal blocks [4]. The four-point functions can be decomposed into conformal partial waves which describe the exchange of primary operators together with their descendants. As for the SCFTs, it can be shown from superconformal algebra that a superconformal primary multiplet can be decomposed into (finite) many conformal primary multiplets, consequently the superconformal block is the summation of several conformal blocks with coefficients restricted by supersymmetry. Previous results on 4D superconformal blocks have been presented in [6, 26–30] based on the superconformal Casimir approach. These studies are mainly focused on the four-point functions of chiral-antichiral fields or conserved currents, which are protected by short-conditions or symmetries. For the four-point functions of more general fields, the traditional superconformal Casimir approach becomes less helpful due to the complex superconformal invariants appearing in the superconformal blocks. Recently a new covariant approach based on the supershadow formalism has been proposed in [31] and applied in [32] for $\mathcal{N} = 1$ superconformal blocks corresponding to exchange of operators neutral under the $U(1)_R$ symmetry.

The new covariant approach generalizes the embedding and shadow formalisms proposed for CFTs to treat with supersymmetric theories. The embedding formalism [33–39] realizes

conformal transformations linearly and provides a convenient way to construct conformally covariant correlation functions. Specifically, the conformal covariance of correlation function is mapped into Lorentz covariance of the correlation function in embedding space. Recently the embedding formalism has been widely used to study the conformal blocks of spinor or tensor operators [21, 25, 40–44]. The $SU(2, 2|\mathcal{N})$ superconformal symmetry transformations can be linearly realized in the supersymmetric generalization–superembedding space [45–49]. The shadow formalism was first proposed in [50–52] and recently applied in computing conformal blocks [39]. Using the shadow operators one can construct projectors of the four-point function which decomposes the four-point function into conformal blocks represented by the exchanged primary operator, actually it provides an analytical method to compute the conformal blocks, and similarly, its supersymmetric generalization gives a systematic method to study the $\mathcal{N} = 1$ superconformal blocks.

In this work we will apply the supershadow formalism to study the most general $\mathcal{N} = 1$ four-point functions of scalars $\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle$, where the scalars Φ_i have independent scaling dimensions and R-charges. The only constraint is from vanishing net R-charges of four scalars so that the $U(1)_R$ symmetry is preserved. Through partial wave decomposition the four-point function gives rise to the most general superconformal blocks, which provide crucial ingredients for $\mathcal{N} = 1$ superconformal bootstrap. Our results are especially important for bootstrapping mixed correlators of scalars with arbitrary scaling dimensions and R-charges, which are beyond previous results on $\mathcal{N} = 1$ superconformal blocks. A rather interesting problem is to bootstrap the mixed correlators between chiral and real scalars which appear in the minimal $4\mathcal{D}$ $\mathcal{N} = 1$ SCFT [7, 53, 54].

This paper is organized as follows. In section 2 we briefly review the superembedding space, supershadow formalism and their roles in computing $\mathcal{N} = 1$ superconformal partial waves. In section 3 we study the most general three-point correlators consisting of two scalars and a spin- ℓ operator with arbitrary scaling dimensions and $U(1)$ R-charges. In section 4 we compute the superconformal partial waves, which are the supershadow projection of the four-point function and obtained from products of two three-point functions. The major difficulty comes from different superconformal weights of scalars, which break the symmetry under coordinate interchange $1 \leftrightarrow 3, 2 \leftrightarrow 4$. Without such symmetry it gets more tricky to evaluate the superconformal integrations. We present the final results on superconformal

blocks in section 5, and compare our general superconformal blocks with known examples as a non-trivial consistent check. Conclusions are made in section 6. We will follow the conventions used in [31, 32] throughout this paper.

II. BRIEF REVIEW OF SUPEREMBEDDING SPACE AND SUPERSHADOW FORMALISM

In this part we briefly review the superembedding space and supershadow formalism, especially for the techniques needed in our computation. More details on these topics are presented in [31, 32, 39, 45, 46].

A. Superembedding Space

There are two equivalent ways to construct superspace in which the $4\mathcal{D}$ $\mathcal{N} = 1$ superconformal group $SU(2, 2|1)$ acts linearly. A natural choice is to construct (anti-) fundamental representation of $SU(2, 2|1)$, the (dual) supertwistor $\mathcal{Y}_A \in \mathbb{C}^{4|1}$ ($\bar{\mathcal{Y}}^A$):

$$\mathcal{Y}_A = \begin{pmatrix} Y_\alpha \\ Y^{\dot{\alpha}} \\ Y_5 \end{pmatrix}, \quad \bar{\mathcal{Y}}^A = \left(\bar{Y}^\alpha \quad \bar{Y}_{\dot{\alpha}} \quad \bar{Y}^5 \right), \quad (1)$$

where Y_α and $Y^{\dot{\alpha}}$ are bosonic complex components while Y_5 is fermionic. Representation for extended supersymmetry $\mathcal{N} > 1$ can be realized with more fermionic components in the supertwistors.

The well-known $4\mathcal{D}$ $\mathcal{N} = 1$ chiral superspace $(x_+^{\dot{\alpha}\alpha}, \theta_i^\alpha)$ can be reproduced from a pair of supertwistors \mathcal{Y}_i^m , $m = 1, 2$, with following constraints

$$\bar{\mathcal{Y}}^{nA} \mathcal{Y}_A^m = 0, \quad m, n = 1, 2. \quad (2)$$

Here one needs to fix the $GL(2, \mathbb{C})$ gauge redundancy arising from the rotation of the two supertwistors, and similarly for the dual supertwistors. Taking the gauge named ‘‘Poincaré

section”, the supertwistor and its dual are simplified into

$$\mathcal{Y}_A^m = \begin{pmatrix} \delta_\alpha^m \\ ix_+^{\dot{\alpha}m} \\ 2\theta^m \end{pmatrix}, \quad \bar{\mathcal{Y}}^{nA} = \begin{pmatrix} -ix_-^{n\alpha} & \delta_\alpha^n & 2\bar{\theta}^n \end{pmatrix}. \quad (3)$$

In the “Poincaré section” the constraints (2) turn into $x_+ - x_- - 4i\bar{\theta}\theta = 0$ and can be solved by the chiral-antichiral coordinates of $4\mathcal{D}\mathcal{N} = 1$ superspace.

The superembedding space provides another way to realize superconformal transformations linearly. Its coordinates are bi-supertwistors $(\mathcal{X}, \bar{\mathcal{X}})$

$$\mathcal{X}_{AB} \equiv \mathcal{Y}_A^m \mathcal{Y}_B^n \epsilon_{mn}, \quad \bar{\mathcal{X}}^{AB} \equiv \bar{\mathcal{Y}}^{iA} \bar{\mathcal{Y}}^{jB} \epsilon_{ij}, \quad (4)$$

By construction, the bi-supertwistors are invariant under $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C})$ and significantly reduce the gauge redundancies of supertwistors, besides, they satisfy the “null” conditions

$$\bar{\mathcal{X}}^{AB} \mathcal{X}_{BC} = 0. \quad (5)$$

Superconformal invariants are obtained from superstraces of successive products of \mathcal{X} ’s and $\bar{\mathcal{X}}$ ’s. For example, the two-point invariant $\langle \bar{2}1 \rangle \equiv \text{Tr}(\bar{\mathcal{X}}_2 \mathcal{X}_1)$ ¹ is

$$\langle \bar{2}1 \rangle \equiv \bar{\mathcal{X}}_2^{AB} \mathcal{X}_{1BA} = -2(x_{2-} - x_{1+} + 2i\theta_1 \sigma \bar{\theta}_2)^2, \quad (6)$$

where the last step is evaluated in the Poincaré section and it is easy to show that

$$\langle \bar{2}1 \rangle^\dagger = \langle \bar{1}2 \rangle. \quad (7)$$

The $\mathcal{N} = 1$ superconformal multiplets can be directly lifted to superembedding space. There are four parameters to characterize a $4\mathcal{D}\mathcal{N} = 1$ superconformal primary superfield \mathcal{O} : the $\text{SL}(2, \mathbb{C})$ Lorentz quantum numbers $(\frac{\ell}{2}, \frac{\bar{\ell}}{2})$, the scaling dimension Δ and $U(1)_R$ charge $R_{\mathcal{O}}$. For SCFTs, usually it is more convenient to use superconformal weights q, \bar{q}

$$q \equiv \frac{1}{2} \left(\Delta + \frac{3}{2} R_{\mathcal{O}} \right), \quad \bar{q} \equiv \frac{1}{2} \left(\Delta - \frac{3}{2} R_{\mathcal{O}} \right) \quad (8)$$

¹ Here and after the indices (j, \bar{k}, \dots) denote the superembedding variables $(\mathcal{X}_j, \bar{\mathcal{X}}_k, \dots)$.

rather than the scaling dimension Δ . Given a superfield $\phi_{\alpha_1 \dots \alpha_\ell}^{\dot{\beta}_1 \dots \dot{\beta}_{\bar{\ell}}} : (\frac{\ell}{2}, \frac{\bar{\ell}}{2}, q, \bar{q})$, its map in superembedding space is a multi-twistor $\Phi_{B_1 \dots B_{\bar{\ell}}}^{A_1 \dots A_\ell}(\mathcal{X}, \bar{\mathcal{X}})$ with homogeneity

$$\Phi(\lambda \mathcal{X}, \bar{\lambda} \bar{\mathcal{X}}) = \lambda^{-q - \frac{\ell}{2}} \bar{\lambda}^{-\bar{q} - \frac{\bar{\ell}}{2}} \Phi(\mathcal{X}, \bar{\mathcal{X}}). \quad (9)$$

The twistor indices make the computations cumbersome, especially for operators with large spin ℓ . Such difficulty is overcome in [37] based on an index-free notation for non-supersymmetric CFTs. The index-free notation is further generalized for $\mathcal{N} = 1$ 4D SCFTs in [31]. The authors introduced pairs of null auxiliary twistors $\mathcal{S}_A, \bar{\mathcal{S}}^A : \bar{\mathcal{S}}^A \mathcal{S}_A = 0$, which are used to contract with twistor indices of lifted fields

$$\Phi(\mathcal{X}, \bar{\mathcal{X}}, \mathcal{S}, \bar{\mathcal{S}}) \equiv \bar{\mathcal{S}}^{B_{\bar{\ell}}} \dots \bar{\mathcal{S}}^{B_1} \Phi_{B_1 \dots B_{\bar{\ell}}}^{A_1 \dots A_\ell} \mathcal{S}_{A_\ell} \dots \mathcal{S}_{A_1}. \quad (10)$$

As construction, $\Phi(\mathcal{X}, \bar{\mathcal{X}}, \mathcal{S}, \bar{\mathcal{S}})$ is a polynomial of $\mathcal{S}_A, \bar{\mathcal{S}}^A$ while with no tensor index, and conversely, one can reproduce the initial superfield from the index-free superembedding fields $\Phi(\mathcal{X}, \bar{\mathcal{X}}, \mathcal{S}, \bar{\mathcal{S}})$ through

$$\phi_{\alpha_1 \dots \alpha_\ell}^{\dot{\beta}_1 \dots \dot{\beta}_{\bar{\ell}}} = \frac{1}{\ell!} \frac{1}{\bar{\ell}!} \left(\bar{\mathcal{X}} \vec{\partial}_{\bar{\mathcal{S}}} \right)^{\dot{\beta}_1} \dots \left(\bar{\mathcal{X}} \vec{\partial}_{\bar{\mathcal{S}}} \right)^{\dot{\beta}_{\bar{\ell}}} \Phi(\mathcal{X}, \bar{\mathcal{X}}, \mathcal{S}, \bar{\mathcal{S}}) \left(\overleftarrow{\partial}_{\mathcal{S}} \mathcal{X} \right)_{\alpha_1} \dots \left(\overleftarrow{\partial}_{\mathcal{S}} \mathcal{X} \right)_{\alpha_\ell} \Big|_{\text{Poincaré}}. \quad (11)$$

To fix gauge redundancies in the lifted fields the auxiliary fields are set to be transverse $\bar{\mathcal{X}} \mathcal{S} = 0, \bar{\mathcal{S}} \mathcal{X} = 0$.

Strings with auxiliary fields, like $\bar{\mathcal{S}}_i j \bar{k} l \dots \bar{m} \mathcal{S}_n$ are superconformal invariant so provide a new type of superconformal invariants besides the supertraces of superembedding coordinates. Correlation functions are built from the two kinds of superconformal invariants. In particular, the two-point function can be completely determined by imposing homogeneity conditions.

It gets more tricky in evaluating three-point functions $\langle \Phi_1(1, \bar{1}) \Phi_2(2, \bar{2}) \Phi_3(3, \bar{3}) \rangle$. For nonsupersymmetric CFTs, conformal symmetry and homogeneities of lifted fields are sufficient to fix three-point functions up to a constant. While for SCFTs, the degree of freedoms of superembedding coordinates are notably enlarged by fermionic components, and it is possible to construct superconformal invariant cross ratio even for three-point correlator, in contrast in CFTs it is impossible to construct conformal invariant cross ratio with fields less than 4. The invariant cross ratio is built from supertraces [46, 55, 56]

$$u = \frac{\langle 1\bar{2} \rangle \langle 2\bar{3} \rangle \langle 3\bar{1} \rangle}{\langle 2\bar{1} \rangle \langle 3\bar{2} \rangle \langle 1\bar{3} \rangle}, \quad (12)$$

which has no contribution on the homogeneity. In consequence, the three-point function can be arbitrary function of the cross ratio u . Denoting

$$z = \frac{1-u}{1+u}, \quad (13)$$

one can show that z is proportional to the fermionic components θ_i , $\bar{\theta}_i$ and satisfies

$$z^3 = 0, \quad z|_{1 \leftrightarrow 2} = z^\dagger = -z. \quad (14)$$

Therefore the most general function of z appearing in the three-point function is up to the second order, besides, considering its symmetry property under permutation $1 \leftrightarrow 2$, there are four free parameters in the general three-point functions [32]. Additional restrictions, like chirality can provide strong constraints on the parameters and simplify the three-point functions drastically. More details on the three-point correlators of general scalars will be studied in Section 3.

B. Supershadow Formalism

The supershadow approach is based on the observation that two operators $\mathcal{O} : (\frac{\ell}{2}, \frac{\bar{\ell}}{2}, q, \bar{q})$ and $\tilde{\mathcal{O}} : (\frac{\bar{\ell}}{2}, \frac{\ell}{2}, 1-q, 1-\bar{q})$ share the same superconformal Casimir so have non-vanishing two-point function. Then the operator $\tilde{\mathcal{O}}$, which is referred to shadow operator of \mathcal{O} , can be used to project the correlation functions onto irreducible representation of \mathcal{O} , i.e., the superconformal partial wave corresponding to exchange primary field \mathcal{O} and its descendants.

The shadow operator $\tilde{\mathcal{O}}$ can be constructed from \mathcal{O} through

$$\tilde{\mathcal{O}}(1, \bar{1}, \mathcal{S}, \bar{\mathcal{S}}) \equiv \int D[2, \bar{2}] \frac{\mathcal{O}^\dagger(2, \bar{2}, 2\bar{\mathcal{S}}, 2\mathcal{S})}{\langle 1\bar{2} \rangle^{1-q+\frac{\ell}{2}} \langle \bar{1}2 \rangle^{1-\bar{q}+\frac{\bar{\ell}}{2}}}, \quad (15)$$

where $D[2, \bar{2}]$ gives the superconformal measure. One can show that the operator obtained from (15) has the expected quantum numbers of shadow operator $\tilde{\mathcal{O}}$. Then it is straightforward to write down the projector

$$|\mathcal{O}| = \frac{1}{\ell!^2 \bar{\ell}!^2} \int_M D[1, \bar{1}] \mathcal{O}(1, \bar{1}, \mathcal{S}, \bar{\mathcal{S}}) \left(\overleftarrow{\partial}_{\mathcal{S}} 1 \overrightarrow{\partial}_{\mathcal{T}} \right)^\ell \left(\overleftarrow{\partial}_{\bar{\mathcal{S}}} \bar{1} \overrightarrow{\partial}_{\bar{\mathcal{T}}} \right)^{\bar{\ell}} \langle \tilde{\mathcal{O}}(1, \bar{1}, \mathcal{T}, \bar{\mathcal{T}}) \rangle, \quad (16)$$

in which the denotation M indicates “monodromy projection” [39]. By inserting the projector $|\mathcal{O}|$ into the four-point function $\langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle$ one can get the superconformal partial

wave $\mathcal{W}_{\mathcal{O}}$

$$\mathcal{W}_{\mathcal{O}} \propto \langle \Phi_1 \Phi_2 | \mathcal{O} | \Phi_3 \Phi_4 \rangle, \quad (17)$$

which corresponds to exchange \mathcal{O} and its descendants. Here the supershadow projector reduces the four-point function into a product of two three-point functions $\langle \Phi_1 \Phi_2 \mathcal{O} \rangle$ and $\langle \tilde{\mathcal{O}} \Phi_3 \Phi_4 \rangle$, which as discussed before, can be easily obtained from superembedding formalism.

The remaining problem is to evaluate the integration in superembedding space. Normally the integrations involve in both bosonic and fermionic components and are rather complex, while for the scalar four-point functions, where the external fermionic components of Φ_i are vanished $\theta_i \equiv \theta_{\text{ext}} = 0$, it was proved in [31] that the integrations can be simplified into non-supersymmetric cases

$$\int D[\mathcal{Y}, \bar{\mathcal{Y}}] g(\mathcal{X}, \bar{\mathcal{X}})|_{\theta_{\text{ext}}=\bar{\theta}_{\text{ext}}=0} = \int D^4 X \partial_{\bar{X}}^2 g(X, \bar{X})|_{\bar{X}=X}, \quad (18)$$

where the embedding coordinates X 's are the bosonic part of superembedding coordinates \mathcal{X} 's. Right hand side integration in embedding space has been comprehensively studied in [39].

Combining all these materials together one can study the $\mathcal{N} = 1$ superconformal blocks analytically and the results can be expressed in a compact form. Superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ for real ($U(1)_R$ neutral) \mathcal{O} has been studied in [32]. In the following part we will apply this method to solve the most general superconformal partial waves.

III. GENERAL THREE-POINT FUNCTIONS

In this section we analyze the most general three-point function $\langle \Phi_1(1, \bar{1}) \Phi_2(2, \bar{2}) \mathcal{O}(0, \bar{0}) \rangle$. The scalars Φ_1, Φ_2 have independent superconformal weights (q_1, \bar{q}_1) and (q_2, \bar{q}_2) , respectively. The exchanged superprimary operator \mathcal{O} has quantum numbers $(\frac{\ell}{2}, \frac{\ell}{2}, \Delta, R_{\mathcal{O}})$, where its $U(1)_R$ charge is $R_{\mathcal{O}} = \frac{2}{3}R \equiv \frac{2}{3}(\bar{q}_1 + \bar{q}_2 - q_1 - q_2)$. From superembedding coordinates we can construct superconformal invariants $\langle ij \bar{j} \rangle$ with $i, j \in 0, 1, 2$, two elementary tensor structures

$$S \equiv \frac{\bar{S}1\bar{2}\mathcal{S}}{\langle 1\bar{2} \rangle}, \quad S|_{1 \leftrightarrow 2} = S^\dagger \equiv \frac{\bar{S}2\bar{1}\mathcal{S}}{\langle 2\bar{1} \rangle}, \quad (19)$$

and also the invariant cross ratio z . For superprimary operators \mathcal{O} with spin- ℓ , it is useful to construct following “eigen” tensor structures with parity $\pm(-1)^\ell$ under coordinate

interchange $1 \leftrightarrow 2$:

$$\begin{aligned} S_-^\ell &= \frac{1}{2} (S^\ell + (-1)^\ell (1 \leftrightarrow 2)), \\ S_+ S_-^{\ell-1} &= \frac{1}{2\ell} (S^\ell - (-1)^\ell (1 \leftrightarrow 2)). \end{aligned} \quad (20)$$

All the spin- ℓ tensor structures $S_+^m S_-^{\ell-m}$ with $m \geq 2$ vanish due to the null condition of S_+ .

The most general three-point function is constructed in terms of supertraces, invariant cross ratio and tensor structures as follows:

$$\begin{aligned} \langle \Phi_1(1, \bar{1}) \Phi_2(2, \bar{2}) \mathcal{O}(0, \bar{0}, \mathcal{S}, \bar{\mathcal{S}}) \rangle &= \\ &= \frac{\left(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)} + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)} z + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(2)} z^2 \right) S_-^\ell + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)} S_+ S_-^{\ell-1}}{(\langle 1\bar{0} \rangle \langle 2\bar{0} \rangle)^\delta \langle 1\bar{2} \rangle^{q_1-\delta} \langle 2\bar{1} \rangle^{q_2-\delta} \langle 0\bar{2} \rangle^{(\bar{q}_2-q_1)+\delta} \langle 0\bar{1} \rangle^{(\bar{q}_1-q_2)+\delta}}, \end{aligned} \quad (21)$$

where $\delta \equiv \frac{1}{4}(\Delta + \ell - R)$. The numerator contains four free coefficients according to the properties of spin- ℓ tensor structures and invariant cross ratio z . It is straightforward to show that the denominator satisfies the homogeneity conditions of the three operators, but this is not the only choice. The homogeneity conditions can only fix the powers of supertraces $\langle i\bar{j} \rangle$ up to a free parameter. Specifically, one can adjust the powers of supertraces through the identity

$$\left(\frac{\langle 1\bar{2} \rangle}{\langle 1\bar{0} \rangle \langle 0\bar{2} \rangle} \right)^{2a} = \left(\frac{\langle 1\bar{2} \rangle \langle 2\bar{1} \rangle}{\langle 1\bar{0} \rangle \langle 0\bar{2} \rangle \langle 0\bar{1} \rangle \langle 2\bar{0} \rangle} \right)^a (1 - 2az + 2a^2 z^2), \quad (22)$$

in the meanwhile, the coefficients $\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)}$ will be transformed linearly. In (21) we have adopted a particular gauge that the supertraces $\langle 1\bar{0} \rangle$ and $\langle 2\bar{0} \rangle$ have identical power. It will be more convenient to compute superconformal integration in this gauge.

A. Remarks on the Complex Coefficients

For the three-point correlator of scalars with arbitrary superconformal weights, it needs to clarify the relationship between $(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)})^*$ and $\lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(i)}$.

Let us evaluate three-point correlator $\langle \Phi_2^\dagger(1, \bar{1}) \Phi_1^\dagger(2, \bar{2}) \mathcal{O}^\dagger(0, \bar{0}) \rangle$. We can directly apply Eq. (21) with three group of quantum numbers $(0, 0, \bar{q}_2, q_2), (0, 0, \bar{q}_1, q_1), (\frac{\ell}{2}, \frac{\ell}{2}, \Delta, -R_{\mathcal{O}})$:

$$\begin{aligned} \langle \Phi_2^\dagger(1, \bar{1}) \Phi_1^\dagger(2, \bar{2}) \mathcal{O}^\dagger(0, \bar{0}) \rangle &= \\ &= \frac{\left(\lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(0)} + \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(1)} z + \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(2)} z^2 \right) S_-^\ell + \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(3)} S_+ S_-^{\ell-1}}{(\langle 1\bar{0} \rangle \langle 2\bar{0} \rangle)^{\delta'} \langle 1\bar{2} \rangle^{\bar{q}_2-\delta'} \langle 2\bar{1} \rangle^{\bar{q}_1-\delta'} \langle 0\bar{2} \rangle^{(q_1-\bar{q}_2)+\delta'} \langle 0\bar{1} \rangle^{(q_2-\bar{q}_1)+\delta'}}, \end{aligned} \quad (23)$$

where $\delta' \equiv \frac{1}{4}(\Delta + \ell + R)$.

Alternatively, we can also solve above three-point correlator by taking Hermitian conjugate on (21) and then permuting coordinates $1 \leftrightarrow 2$. Both the invariant cross ratio z and the spin- ℓ tensor structure S are invariant under the combination actions of Hermitian conjugate and coordinate permutation $1 \leftrightarrow 2$, the new three-point function turns into

$$\langle \Phi_1^\dagger(2, \bar{2}) \Phi_2^\dagger(1, \bar{1}) \mathcal{O}^\dagger(0, \bar{0}) \rangle = \frac{\left((\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)})^* + (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)})^* z + (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(2)})^* z^2 \right) S_-^\ell + (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)})^* S_+ S_-^{\ell-1}}{(\langle 0\bar{1} \rangle \langle 0\bar{2} \rangle)^\delta \langle 1\bar{2} \rangle^{q_1-\delta} \langle 2\bar{1} \rangle^{q_2-\delta} \langle 1\bar{0} \rangle^{(\bar{q}_2-q_1)+\delta} \langle 2\bar{0} \rangle^{(\bar{q}_1-q_2)+\delta}}. \quad (24)$$

To compare Eq. (24) with Eq. (23), we need to make a transformation (22) in Eq. (24) with parameter

$$a = \frac{q_2 + \bar{q}_2 - q_1 - \bar{q}_1}{2} = -\frac{r}{2}, \quad (25)$$

then the two equations share exactly the same denominator. Identifying the tensor structures in their numerators, we obtain following linear relationships among the complex coefficients

$$\begin{aligned} \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(0)} &= (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)})^*, \\ \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(1)} &= r(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)})^* + (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)})^*, \\ \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(2)} &= \frac{1}{2}r^2(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)})^* + r(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)})^* + (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(2)})^* + \frac{1}{2}r(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)})^*, \\ \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(3)} &= (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)})^*. \end{aligned} \quad (26)$$

By taking above complex conjugate transformation of the coefficients twice, we go back to the original coefficients, as expected. The linear transformation turns into trivial $(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)})^* = \lambda_{\Phi_2^\dagger \Phi_1^\dagger \mathcal{O}^\dagger}^{(i)}$ given $r = 0$, i.e., scalars Φ_1 and Φ_2 share the same scaling dimension.

B. Three-point Functions with Chiral Operator

Three-point function can be significantly simplified if there is a chiral or anti-chiral operator. Results obtained from these short multiplets will provide key elements to compute the most general superconformal blocks.

Let us consider the three-point correlator $\langle \Phi(1) X(2, \bar{2}) \mathcal{O}(0, \bar{0}) \rangle$ which will be needed to compute the shadow coefficients. The three-point correlator contains a chiral field Φ :

$(0, 0, q_1, 0)$, a general field $X : (0, 0, q_2, \bar{q}_2)$ and a spin- ℓ operator $\mathcal{O} : (\frac{\ell}{2}, \frac{\ell}{2}, \frac{\Delta+R}{2}, \frac{\Delta-R}{2})$, where $R = \bar{q}_2 - q_1 - q_2$. From the chirality of Φ , we can obtain the simplified three-point function

$$\langle \Phi(1)X(2, \bar{2})\mathcal{O}(0, \bar{0}, \mathcal{S}, \bar{\mathcal{S}}) \rangle = \frac{\lambda_{\Phi X \mathcal{O}} S^\ell}{\langle 1\bar{2} \rangle^{\frac{1}{2}(q_1+q_2+\bar{q}_2-\Delta-\ell)} \langle 1\bar{0} \rangle^{\frac{1}{2}(q_1-q_2-\bar{q}_2+\Delta+\ell)} \langle 2\bar{0} \rangle^{q_2} \langle 0\bar{2} \rangle^{\frac{1}{2}(-q_1-q_2+\bar{q}_2+\Delta+\ell)}}. \quad (27)$$

Taking the transformation (22) with $a = \frac{1}{4}(\Delta + \ell + 2r + R)$, where $r = q_1 - q_2 - \bar{q}_2$ in this problem, above equation turns into

$$\begin{aligned} \langle \Phi(1)X(2, \bar{2})\mathcal{O}(0, \bar{0}, \mathcal{S}, \bar{\mathcal{S}}) \rangle &= \\ &= \lambda_{\Phi X \mathcal{O}} \frac{(1 - 2az + a(2a - \ell)z^2)S_-^\ell + \ell S_+ S_-^{\ell-1}}{\langle 1\bar{2} \rangle^{q_1-q_2-a} \langle 2\bar{1} \rangle^{-a} (\langle 1\bar{0} \rangle \langle 2\bar{0} \rangle)^{a+q_2} \langle 0\bar{1} \rangle^a \langle 0\bar{2} \rangle^{a+q_2+\bar{q}_2-q_1}}, \end{aligned} \quad (28)$$

which is consistent with the general three-point function (21) given $\bar{q}_1 = 0$, $\delta = a + q_2$. The four free coefficients are fixed by the chirality condition up to an overall constant. Such kind of three-point function with real X appears in bootstrapping the mixed correlator of minimal $4D$ $N = 1$ SCFT. In the theory the scalar X appears in OPE $\Phi \times \Phi^\dagger$ so is real: $q_2 = \bar{q}_2$.

Similarly, one can use anti-chirality condition to partially fix the coefficients in three-point function $\langle \Phi(\bar{1})^\dagger X(2, \bar{2})\mathcal{O}(0, \bar{0}) \rangle$:

$$(\lambda_{\Phi^\dagger X \mathcal{O}}^{(0)}, \lambda_{\Phi^\dagger X \mathcal{O}}^{(2)}, \lambda_{\Phi^\dagger X \mathcal{O}}^{(1)}, \lambda_{\Phi^\dagger X \mathcal{O}}^{(3)}) = \lambda_{\Phi^\dagger X \mathcal{O}}(1, a'(2a' - \ell), -2a', \ell), \quad (29)$$

where $a' = \frac{1}{4}(\Delta + \ell - R)$, $R = \bar{q}_1 + \bar{q}_2 - q_2$.

IV. SUPERCONFORMAL PARTIAL WAVES

Now we are ready to study the most general four-point correlator

$$\langle \Phi_1(1, \bar{1})\Phi_2(2, \bar{2})\Phi_3(3, \bar{3})\Phi_4(4, \bar{4}) \rangle, \quad (30)$$

where Φ_i have arbitrary superconformal weights (q_i, \bar{q}_i) constrained by vanishing net R-charges

$$\sum_i q_i - \sum_i \bar{q}_i = 0. \quad (31)$$

Here we are interested in the superconformal partial wave which gives the amplitude of exchanging an irreducible representation of the $\mathcal{N} = 1$ superconformal group. Let us denote such irreducible representation by its superprimary field $\mathcal{O} : (\frac{\ell}{2}, \frac{\ell}{2}, \Delta, R_{\mathcal{O}})$. By inserting the projector constructed from \mathcal{O} and its shadow operator $\tilde{\mathcal{O}}$ into the four-point correlator, the superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ becomes

$$\begin{aligned} \mathcal{W}_{\mathcal{O}} &\propto \langle \Phi_1 \Phi_2 | \mathcal{O} | \Phi_3 \Phi_4 \rangle = \int D[0, \bar{0}] \langle \Phi_1 \Phi_2 \mathcal{O}(0, \bar{0}, \mathcal{S}, \bar{\mathcal{S}}) \rangle \overleftrightarrow{\mathcal{D}}_{\ell} \langle \tilde{\mathcal{O}}(0, \bar{0}, \mathcal{T}, \bar{\mathcal{T}}) \Phi_3 \Phi_4 \rangle \\ &= \frac{1}{\langle 1\bar{2} \rangle^{q_1 - \delta} \langle 2\bar{1} \rangle^{q_2 - \delta} \langle 3\bar{4} \rangle^{q_3 - \delta'} \langle 4\bar{3} \rangle^{q_4 - \delta'}} \times \\ &\quad \int D[0, \bar{0}] \frac{\mathcal{N}_{\ell}^f}{(\langle 1\bar{0} \rangle \langle 2\bar{0} \rangle)^{\delta} (\langle 3\bar{0} \rangle \langle 4\bar{0} \rangle)^{\delta'} \langle 0\bar{2} \rangle^{\delta + \bar{q}_2 - q_1} \langle 0\bar{1} \rangle^{\delta + \bar{q}_1 - q_2} \langle 0\bar{4} \rangle^{\delta' + \bar{q}_4 - q_3} \langle 0\bar{3} \rangle^{\delta' + \bar{q}_3 - q_4}}, \end{aligned} \quad (32)$$

where $\delta = \frac{\Delta + \ell - R}{4}$, $\delta' = \frac{2 + R + \ell - \Delta}{4}$ and $\overleftrightarrow{\mathcal{D}}_{\ell} \equiv \frac{1}{\ell!^4} (\partial_S 0 \partial_{\mathcal{T}})^{\ell} (\partial_{\bar{S}} \bar{0} \partial_{\bar{\mathcal{T}}})^{\ell}$. \mathcal{N}_{ℓ}^f represents the tensor structures as defined in [32]:

$$\begin{aligned} \mathcal{N}_{\ell}^f &= \left((\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)} + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)} z + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(2)} z^2) S_{-}^{\ell} + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)} S_{+} S_{-}^{\ell-1} \right) \\ &\quad \overleftrightarrow{\mathcal{D}}_{\ell} \left((\lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(0)} + \lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(1)} \tilde{z} + \lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(2)} \tilde{z}^2) T_{-}^{\ell} + \lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(3)} T_{+} T_{-}^{\ell-1} \right), \end{aligned} \quad (33)$$

In (33) we have applied the three-point function

$$\begin{aligned} \langle \Phi_3(3, \bar{3}) \Phi_4(4, \bar{4}) \tilde{\mathcal{O}}(0, \bar{0}, \mathcal{T}, \bar{\mathcal{T}}) \rangle &= \\ &\quad \frac{\left(\lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(0)} + \lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(1)} \tilde{z} + \lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(2)} \tilde{z}^2 \right) T_{-}^{\ell} + \lambda_{\Phi_3 \Phi_4 \tilde{\mathcal{O}}}^{(3)} T_{+} T_{-}^{\ell-1}}{(\langle 3\bar{0} \rangle \langle 4\bar{0} \rangle)^{\delta'} \langle 3\bar{4} \rangle^{q_3 - \delta'} \langle 4\bar{3} \rangle^{q_4 - \delta'} \langle 0\bar{4} \rangle^{(\bar{q}_4 - q_3) + \delta'} \langle 0\bar{3} \rangle^{(\bar{q}_3 - q_4) + \delta'}}. \end{aligned} \quad (34)$$

where $(\tilde{z}, T_{\pm}^{\ell})$, like (z, S_{\pm}^{ℓ}) in (21), are invariant cross ratio and tensor structures. Tensor structures in \mathcal{N}_{ℓ}^f consist of the polynomial \mathcal{N}_{ℓ}

$$\mathcal{N}_{\ell} \equiv (\bar{\mathcal{S}} 1 \bar{2} \mathcal{S})^{\ell} \overleftrightarrow{\mathcal{D}}_{\ell} (\bar{\mathcal{T}} 3 \bar{4} \mathcal{T})^{\ell} \quad (35)$$

and its coordinate exchanges. Giving $\theta_{\text{ext}} = \bar{\theta}_{\text{ext}} = 0$ and $\mathcal{X}_0 = \bar{\mathcal{X}}_0$, \mathcal{N}_{ℓ} reduces to

$$N_{\ell} = y_0^{\frac{\ell}{2}} C_{\ell}^{(1)}(y_0), \quad (36)$$

where $C_{\ell}^{(\lambda)}(y)$ are the Gegenbauer polynomials and

$$x_0 \equiv -\frac{X_{13} X_{20} X_{40}}{2\sqrt{X_{10} X_{20} X_{30} X_{40} X_{12} X_{34}}} - (1 \leftrightarrow 2) - (3 \leftrightarrow 4), \quad (37)$$

$$y_0 \equiv \frac{1}{2^{12}} X_{10} X_{20} X_{30} X_{40} X_{12} X_{34}. \quad (38)$$

For the four-point function of scalars we are only interested in the lowest component of a supermultiplet. To throw away irrelevant higher dimensional components we set the fermionic coordinates $\theta_{\text{ext}} = \bar{\theta}_{\text{ext}} = 0$. The bi-supertwistors $\mathcal{X}_{AB}, \tilde{\mathcal{X}}^{AB}$ degenerate into twistors $X_{\alpha\beta}, X^{\alpha\beta}$ which are equivalent to the six dimensional vector representations of $\text{SU}(2, 2) \cong \text{SO}(4, 2)$, and the supertraces $\langle i\bar{j} \rangle$ become inner products of vectors $X_{ij} \equiv -2X_i \cdot X_j$. Moreover, under the restriction $\theta_{\text{ext}} = \bar{\theta}_{\text{ext}} = 0$ the superconformal integration (33) can be simplified into nonsupersymmetric conformal integration, as suggested in (18). To summarize, the superconformal partial wave $\mathcal{W}_{\mathcal{O}}$ is

$$\mathcal{W}_{\mathcal{O}}|_{\theta_{\text{ext}}=0} \propto \frac{1}{X_{12}^{q_1+q_2-2\delta} X_{34}^{q_3+q_4-2\delta'}} \int D^4 X_0 \partial_0^2 \frac{\mathcal{N}_{\ell}^f}{D_{\ell}} \Big|_{\bar{0}=0}, \quad (39)$$

and D_{ℓ} denotes the products of supertraces containing \mathcal{X}_0 or $\bar{\mathcal{X}}_0$

$$D_{\ell} \equiv (X_{1\bar{0}} X_{2\bar{0}})^{\delta} (X_{3\bar{0}} X_{4\bar{0}})^{\delta'} X_{0\bar{2}}^{\delta+\bar{q}_2-q_1} X_{0\bar{1}}^{\delta+\bar{q}_1-q_2} X_{0\bar{4}}^{\delta'+\bar{q}_4-q_3} X_{0\bar{3}}^{\delta'+\bar{q}_3-q_4}. \quad (40)$$

As shown in (39), essentially there are only two steps to accomplish the superconformal integration for $\mathcal{W}_{\mathcal{O}}$: partial derivatives on $\mathcal{N}_{\ell}^f/D_{\ell}$ and conformal integration. The partial derivatives are straightforward to evaluate. The conformal integration related to Gegenbauer polynomial $C_{\ell}^{(1)}(x_0)$ has been detailedly studied in [39, 57]. Since the result is fundamental for our study we repeat it here for convenience

$$\int_M D^4 X_0 \frac{(-1)^{\ell} C_{\ell}^{(1)}(x_0)}{X_{10}^{\frac{\Delta+r}{2}} X_{20}^{\frac{\Delta-r}{2}} X_{30}^{\frac{\tilde{\Delta}+\tilde{r}}{2}} X_{40}^{\frac{\tilde{\Delta}-\tilde{r}}{2}}} = \xi_{\Delta, \tilde{\Delta}, \tilde{r}, \ell} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} X_{12}^{-\frac{\Delta}{2}} X_{34}^{-\frac{\tilde{\Delta}}{2}} g_{\Delta, \ell}^{r, \tilde{r}}(u, v), \quad (41)$$

in which $r \equiv \Delta_1 - \Delta_2$, $\tilde{r} \equiv \Delta_3 - \Delta_4$ and

$$\xi_{\Delta, \tilde{\Delta}, \tilde{r}, \ell} \equiv \frac{\pi^2 \Gamma(\tilde{\Delta} + \ell - 1) \Gamma(\frac{\Delta - \tilde{r} + \ell}{2}) \Gamma(\frac{\Delta + \tilde{r} + \ell}{2})}{(2 - \Delta) \Gamma(\Delta + \ell) \Gamma(\frac{\tilde{\Delta} - \tilde{r} + \ell}{2}) \Gamma(\frac{\tilde{\Delta} + \tilde{r} + \ell}{2})}. \quad (42)$$

The conformal blocks $g_{\Delta, \ell}^{r, \tilde{r}}(u, v)$ are defined as usual

$$\begin{aligned} g_{\Delta, \ell}^{r, \tilde{r}}(u, v) &= \frac{\rho \bar{\rho}}{\rho - \bar{\rho}} [k_{\Delta+\ell}(\rho) k_{\Delta-\ell-2}(\bar{\rho}) - (\rho \leftrightarrow \bar{\rho})], \\ k_{\beta}(x) &= x^{\frac{\beta}{2}} {}_2F_1 \left(\frac{\beta-r}{2}, \frac{\beta+\tilde{r}}{2}, \beta, x \right), \end{aligned} \quad (43)$$

where u, v are the standard conformal invariants and $u = \rho \bar{\rho}$, $v = (1 - \rho)(1 - \bar{\rho})$.

To apply above results on conformal integrations in our case, the most crucial step is to write the integrand into a compact form in terms of Gegenbauer polynomials.

Giving $\theta_{\text{ext}} = \bar{\theta}_{\text{ext}} = 0$, the only non-vanishing fermionic coordinates are $\theta_0, \bar{\theta}_0$ from bisupertwistors $\mathcal{X}_0, \bar{\mathcal{X}}_0$. Supconformal invariants proportional to the fermionic coordinates therefore vanish at third or higher orders. Moreover, as shown in [32], the tensor structure terms in \mathcal{N}_ℓ^f can be separated into symmetric (\mathcal{N}_ℓ^+) or antisymmetric (\mathcal{N}_ℓ^-) parts according to their performances under coordinate interchange $1 \leftrightarrow 3, 2 \leftrightarrow 4$:

$$\mathcal{N}_\ell^f = \mathcal{N}_\ell^+ + \mathcal{N}_\ell^-, \quad (44)$$

in which

$$\begin{aligned} \mathcal{N}_\ell^+ = & S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell \left(\lambda_{\Phi_1 \Phi_2}^{(0)} \lambda_{\Phi_3 \Phi_4}^{(0)} \bar{\mathcal{O}} + \lambda_{\Phi_1 \Phi_2}^{(2)} \lambda_{\Phi_3 \Phi_4}^{(0)} \bar{\mathcal{O}} z^2 + \lambda_{\Phi_1 \Phi_2}^{(0)} \lambda_{\Phi_3 \Phi_4}^{(2)} \bar{\mathcal{O}} \tilde{z}^2 \right. \\ & \left. + \lambda_{\Phi_1 \Phi_2}^{(1)} \lambda_{\Phi_3 \Phi_4}^{(1)} \bar{\mathcal{O}} z \tilde{z} \right) + S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_+ T_-^{\ell-1} \lambda_{\Phi_1 \Phi_2}^{(1)} \lambda_{\Phi_3 \Phi_4}^{(3)} \bar{\mathcal{O}} z \\ & + S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell \lambda_{\Phi_1 \Phi_2}^{(3)} \lambda_{\Phi_3 \Phi_4}^{(1)} \bar{\mathcal{O}} \tilde{z} + S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}}_\ell T_+ T_-^{\ell-1} \lambda_{\Phi_1 \Phi_2}^{(3)} \lambda_{\Phi_3 \Phi_4}^{(3)} \bar{\mathcal{O}}, \end{aligned} \quad (45)$$

and

$$\begin{aligned} \mathcal{N}_\ell^- = & z S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell \lambda_{\Phi_1 \Phi_2}^{(1)} \lambda_{\Phi_3 \Phi_4}^{(0)} \bar{\mathcal{O}} + \tilde{z} S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell \lambda_{\Phi_1 \Phi_2}^{(0)} \lambda_{\Phi_3 \Phi_4}^{(1)} \bar{\mathcal{O}} \\ & + S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_+ T_-^{\ell-1} \lambda_{\Phi_1 \Phi_2}^{(0)} \lambda_{\Phi_3 \Phi_4}^{(3)} \bar{\mathcal{O}} + S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell \lambda_{\Phi_1 \Phi_2}^{(3)} \lambda_{\Phi_3 \Phi_4}^{(0)} \bar{\mathcal{O}}. \end{aligned} \quad (46)$$

Contributions of the symmetric terms \mathcal{N}_ℓ^+ on the superconformal partial wave $\mathcal{W}_\mathcal{O}$ have been detailedly studied in [32] under the restrictions

$$q_1 = \bar{q}_2, \quad q_2 = \bar{q}_1, \quad q_3 = \bar{q}_4, \quad q_4 = \bar{q}_3. \quad (47)$$

Under above restrictions the coordinate interchange symmetry in \mathcal{N}_ℓ^+ is further realized in the whole integrand of superconformal partial wave $\mathcal{W}_\mathcal{O}$, and due to this symmetry, it gets much simpler to evaluate contributions on $\mathcal{W}_\mathcal{O}$ from the symmetric terms. While for the most general superconformal partial waves we do not have such restrictions on the superconformal weights, nevertheless, there is a free parameter related to the transformation (22), and we can choose the gauge in which $X_{1\bar{0}}$ ($X_{3\bar{0}}$) and $X_{2\bar{0}}$ ($X_{4\bar{0}}$) have the same power, then it is straightforward to calculate contributions of these terms on $\mathcal{W}_\mathcal{O}$. More details on the calculations are provided in Appendix B.

The major challenge comes from the four terms in \mathcal{N}_ℓ^- which are anti-symmetric under the coordinate interchange $1 \leftrightarrow 3, 2 \leftrightarrow 4$ (anti-symmetric terms). For the cases studied in [32], due to the restrictions (47), D_ℓ is invariant under coordinate interchange $1 \leftrightarrow 3, 2 \leftrightarrow 4$, and

contributions from anti-symmetric terms are cancelled automatically. While for general four-point functions there is no such coordinate interchange symmetry in D_ℓ , and contributions from terms in (46) are proportional to the differences of scaling dimensions r, \tilde{r} .

A. Superconformal Integrations of Anti-symmetric Terms

In this section we evaluate superconformal integrations of the anti-symmetric terms in (46) following the strategy discussed before. However, to apply the conformal integration formulas in (41), we need to figure out relationships between tensor structures in \mathcal{N}_ℓ^- and the Gegenbauer polynomials. For tensor structures in \mathcal{N}_ℓ^+ , the polynomials satisfy coordinate interchange symmetry and can be simplified using Clifford algebra. Nevertheless, for tensor structures in \mathcal{N}_ℓ^- , the polynomials are anti-symmetric under coordinate permutation and the Clifford algebra cannot help to simplify the polynomials directly, instead, we show that these polynomials possesses recursion relations which can be used to determine the superconformal integrations.

The anti-symmetric terms in (46) consist of $\frac{zN_\ell}{D_\ell}$, $\frac{\bar{z}N_\ell}{D_\ell}$, $\frac{N_\ell}{D_\ell}$ and their coordinate exchanges. The partial differentiations are

$$\begin{aligned} \partial_0^2 \frac{zN_\ell}{D_\ell} \Big|_{\bar{0}=0} &= 2\delta' \frac{N_\ell}{D_\ell} \left[\frac{X_{13}}{X_{10}X_{30}} - \frac{X_{23}}{X_{20}X_{30}} + \frac{X_{14}}{X_{10}X_{40}} - \frac{X_{24}}{X_{20}X_{40}} \right] \\ &\quad + \frac{1}{2} \frac{1}{D_\ell} \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \left[\frac{X_{12}}{X_{10}X_{20}} X_{10} (S\bar{2}3\bar{4}T) \right], \end{aligned} \quad (48)$$

$$\begin{aligned} \partial_0^2 \frac{\bar{z}N_\ell}{D_\ell} \Big|_{\bar{0}=0} &= 2\delta \frac{N_\ell}{D_\ell} \left[\frac{X_{13}}{X_{10}X_{30}} + \frac{X_{23}}{X_{20}X_{30}} - \frac{X_{14}}{X_{10}X_{40}} - \frac{X_{24}}{X_{20}X_{40}} \right] \\ &\quad + \frac{1}{2} \frac{1}{D_\ell} \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \left[\frac{X_{34}}{X_{30}X_{40}} X_{30} (S\bar{2}1\bar{4}T) \right], \end{aligned} \quad (49)$$

$$\begin{aligned} \partial_0^2 \frac{N_\ell}{D_\ell} \Big|_{\bar{0}=0} &= -\frac{N_\ell}{D_\ell} \left[4\delta^2 \frac{X_{12}}{X_{10}X_{20}} + 4\delta'^2 \frac{X_{34}}{X_{30}X_{40}} + 4\delta\delta' \left(\frac{X_{13}}{X_{10}X_{30}} + \frac{X_{23}}{X_{20}X_{30}} + \frac{X_{14}}{X_{10}X_{40}} \right. \right. \\ &\quad \left. \left. + \frac{X_{24}}{X_{20}X_{40}} \right) \right] + \frac{1}{2} \frac{1}{D_\ell} \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \left[2\delta \frac{X_{12}}{X_{10}X_{20}} (X_{10}S\bar{2}3\bar{4}T) \right. \\ &\quad \left. + 2\delta' \frac{X_{34}}{X_{30}X_{40}} (X_{30}S\bar{2}1\bar{4}T) \right]. \end{aligned} \quad (50)$$

For the terms proportional to N_ℓ , their conformal integrations can be evaluated directly using Eq. (41), the results are provided in Appendix B. While for extra terms, we need to find their relationships with Gegenbauer polynomials before we can apply Eq. (41). Tensor

structures in (46) can be expanded in terms of \mathcal{N}_ℓ and its coordinate exchanges as

$$S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell = \frac{\mathcal{N}_\ell}{4\langle 1\bar{2}\rangle^\ell \langle 3\bar{4}\rangle^\ell} + (-1)^\ell (1 \leftrightarrow 2) + (-1)^\ell (3 \leftrightarrow 4), \quad (51)$$

$$S_-^\ell \overleftrightarrow{\mathcal{D}}_\ell T_+ T_-^{\ell-1} = \frac{\mathcal{N}_\ell}{4\ell \langle 1\bar{2}\rangle^\ell \langle 3\bar{4}\rangle^\ell} + (-1)^\ell (1 \leftrightarrow 2) - (-1)^\ell (3 \leftrightarrow 4), \quad (52)$$

$$S_+ S_-^{\ell-1} \overleftrightarrow{\mathcal{D}}_\ell T_-^\ell = \frac{\mathcal{N}_\ell}{4\ell \langle 1\bar{2}\rangle^\ell \langle 3\bar{4}\rangle^\ell} - (-1)^\ell (1 \leftrightarrow 2) + (-1)^\ell (3 \leftrightarrow 4), \quad (53)$$

which lead to following polynomial terms in the conformal integrand

$$R_\ell \equiv \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \times \\ (X_{10}S\bar{2}3\bar{4}T + X_{20}S\bar{1}3\bar{4}T - X_{10}S\bar{2}4\bar{3}T - X_{20}S\bar{1}4\bar{3}T), \quad (54)$$

$$P_\ell \equiv \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \times \\ (X_{30}S\bar{2}1\bar{4}T + X_{40}S\bar{2}1\bar{3}T - X_{30}S\bar{1}2\bar{4}T - X_{40}S\bar{1}2\bar{3}T). \quad (55)$$

It is shown in Appendix A that above polynomials satisfy the recursion relations

$$R_\ell = \ell \Delta_A N_{\ell-1} + \frac{1}{2^6} (\ell-1) X_{10} X_{20} X_{34} \Delta_B N_{\ell-2} + y_0 R_{\ell-2}, \quad (56)$$

$$P_\ell = \ell \Delta_B N_{\ell-1} + \frac{1}{2^6} (\ell-1) X_{30} X_{40} X_{12} \Delta_A N_{\ell-2} + y_0 P_{\ell-2}. \quad (57)$$

The conformal integrations related to R_ℓ and P_ℓ are

$$\int D^4 X_0 \frac{X_{12}}{X_{10} X_{20}} \frac{R_\ell}{D_\ell} \Big|_{\bar{0}=0} = \frac{8 c_\ell \xi_{\Delta+2,2-\Delta,1+\tilde{r},\ell-1}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} \times \\ \left[- \frac{4\tilde{r}\Delta(\ell+1)(\Delta-\ell)}{(\Delta-1)(\Delta+\tilde{r}-\ell)(\Delta+\tilde{r}+\ell)} g_{\Delta+1,\ell-1}^{r,\tilde{r}} \right. \\ \left. + \frac{r\ell(\Delta-\ell)(\Delta-\tilde{r}+\ell)}{(\Delta+\ell)(\Delta+\ell+1)(\Delta+\tilde{r}-\ell)} g_{\Delta+2,\ell}^{r,\tilde{r}} \right], \quad (58)$$

$$\int D^4 X_0 \frac{X_{34}}{X_{30} X_{40}} \frac{P_\ell}{D_\ell} \Big|_{\bar{0}=0} = \frac{8 c_\ell \xi_{\Delta,4-\Delta,1+\tilde{r},\ell-1}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} \times \\ \left[- \frac{r(\Delta-2)(\ell+1)(-\Delta+\tilde{r}+\ell+2)(\Delta-\tilde{r}+\ell-2)}{4(\Delta-1)(-\Delta+\ell+1)(-\Delta+\ell+2)(\Delta+\ell-1)} g_{\Delta+1,\ell-1}^{r,\tilde{r}} \right. \\ \left. - \frac{\tilde{r}\ell(\Delta-\tilde{r}+\ell-2)}{(\Delta+\ell-1)(\Delta+\tilde{r}-\ell-2)} g_{\Delta,\ell}^{r,\tilde{r}} \right], \quad (59)$$

where $c_\ell = 2^{-6\ell}$. Above equations can be proved using mathematical induction based on the recursion relations (56) and (57). Conformal integrations in (58) and (59), together with the results presented in Appendix B, provide all the necessary materials to compute the superconformal partial waves $\mathcal{W}_\mathcal{O}$ for general scalars Φ_i . Here we present the final results of

superconformal partial wave (39):

$$\mathcal{W}_{\mathcal{O}} \propto \frac{1}{X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} \times \left(a_1 g_{\Delta,\ell}^{r,\tilde{r}} + a_2 g_{\Delta+1,\ell+1}^{r,\tilde{r}} + a_3 g_{\Delta+1,\ell-1}^{r,\tilde{r}} + a_4 g_{\Delta+2,\ell}^{r,\tilde{r}} \right), \quad (60)$$

in which the coefficients a_i are the abbreviations of following long expressions:

$$a_1 = 2\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} \left[-\delta' \left(1 + 2\delta \frac{(2-\Delta)\tilde{r}^2 - (\ell+2-\Delta)(\Delta+\ell)}{(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)} \right) \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(0)} + \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(2)} \right. \\ \left. + \frac{\tilde{r}((\Delta-2)R + (-\Delta+\ell+2)(\Delta+\ell))}{2(-\Delta+\ell+2)(\Delta+\ell)} \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(1)} + \frac{\tilde{r}(R+\ell+2-\Delta)}{4(-\Delta+\ell+2)} \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(3)} \right], \quad (61)$$

$$a_2 = -\frac{(\Delta-2)(\Delta-\tilde{r}+\ell)(\Delta+\tilde{r}+\ell)}{4(\Delta-1)(\Delta+\ell)(\Delta+\ell+1)} \\ \times \left(\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} + \frac{r(\Delta-R+\ell)}{2(\Delta+\ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} \right) \left(\lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(1)} + \frac{\tilde{r}(R+\ell+2-\Delta)}{2(-\Delta+\ell+2)} \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(0)} \right), \quad (62)$$

$$a_3 = -\frac{(\Delta-2)(\Delta-\tilde{r}-\ell-2)(\Delta+\tilde{r}-\ell-2)}{4(\Delta-1)(-\Delta+\ell+1)(-\Delta+\ell+2)} \\ \times \left(\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} + \frac{\ell+1}{\ell} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(3)} + \frac{r(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} \right) \\ \times \left(\lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(1)} + \frac{\ell+1}{\ell} \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(3)} + \frac{\tilde{r}(\Delta-R+\ell)}{2(\Delta+\ell)} \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(0)} \right), \quad (63)$$

$$a_4 = 2\lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(0)} \frac{(\Delta-2)(-\Delta-\tilde{r}+\ell+2)(-\Delta+\tilde{r}+\ell+2)(\Delta-\tilde{r}+\ell)(\Delta+\tilde{r}+\ell)}{16\Delta(-\Delta+\ell+1)(-\Delta+\ell+2)(\Delta+\ell)(\Delta+\ell+1)} \\ \times \left[-\delta \left(1 - 2\delta' \frac{(r^2\Delta - (\Delta+\ell)(-\Delta+\ell+2))}{(\Delta-1)(-\Delta+\ell+2)(\Delta+\ell)} \right) \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} + \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(2)} \right. \\ \left. + \frac{r(\Delta(-\Delta+R+2) + \ell(\ell+2))}{2(-\Delta+\ell+2)(\Delta+\ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} + \frac{r(\Delta-R+\ell)}{4(\Delta+\ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} \right]. \quad (64)$$

Several interesting properties appear in above long expressions of coefficients a_i . Ignoring the constant term, a_1 and a_4 are related to each other through a transformation

$$\Delta \leftrightarrow 2-\Delta, \quad r \leftrightarrow \tilde{r}, \quad R \leftrightarrow -R, \quad \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(i)} \leftrightarrow \lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(i)}, \quad (65)$$

while a_2 and a_3 are invariant under this transformation. Such symmetry is expected since it corresponds to exchange the roles of operator \mathcal{O} and its supershadow operator $\tilde{\mathcal{O}}$.

V. SUPERCONFORMAL BLOCKS

Conformal blocks are obtained from conformal partial waves by dropping some less interesting factors. The $\mathcal{N} = 1$ superconformal block $\mathcal{G}_{\Delta,\ell}^{r,\tilde{r}}$ is related to the superconformal

partial wave $\mathcal{W}_{\mathcal{O}}$ through

$$\mathcal{G}_{\Delta,\ell}^{r,\bar{r}} = X_{12}^{\frac{\Delta_1+\Delta_2}{2}} X_{34}^{\frac{\Delta_3+\Delta_4}{2}} \left(\frac{X_{24}}{X_{14}} \right)^{-\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{-\frac{\bar{r}}{2}} \mathcal{W}_{\mathcal{O}}. \quad (66)$$

Then applying the results on $\mathcal{W}_{\mathcal{O}}$ (60-64) one can get the superconformal block in terms of $\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(i)}$ and $\lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(i)}$. The supershadow coefficients $\lambda_{\Phi_3\Phi_4\tilde{\mathcal{O}}}^{(i)}$ need to be transformed into the normal coefficients $\lambda_{\Phi_3\Phi_4\mathcal{O}^\dagger}^{(i)}$. In principle, one can solve the transformation between the two types of coefficients by inserting the integral expression of supershadow operator $\tilde{\mathcal{O}}$ (15) in the three-point function $\langle \Phi_3(3, \bar{3})\Phi_4(4, \bar{4})\tilde{\mathcal{O}}(0, \bar{0}, \mathcal{T}, \bar{\mathcal{T}}) \rangle$ (34). However it needs to evaluate a complex superconformal integration to obtain the results. A simpler method is proposed in [32] which applies the unitarity of SCFTs. In this work the unitarity of SCFTs is also employed to solve the transformation of supershadow coefficients.

Giving $\Phi_3 = \Phi_2^\dagger$ and $\Phi_4 = \Phi_1^\dagger$, unitarity of the four-point function $\langle \Phi_1\Phi_2\Phi_2^\dagger\Phi_1^\dagger \rangle$ requires the coefficients a_i (61-64) of four conformal blocks in $\mathcal{G}_{\Delta,\ell}^{r,\bar{r}}$ to be positive. To apply the unitary condition we need to go back to the coefficients $(\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(i)})^*$ rather than use $\lambda_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger}^{(i)}$ directly. At first it is not clear whether there is a linear map connecting $\lambda_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}}^{(i)}$ with $(\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(i)})^*$. Possible transformations among the three types of coefficients are shown in graph as below

$$\begin{array}{ccc} \lambda_{\Phi_2^\dagger\Phi_1^\dagger\tilde{\mathcal{O}}}^{(i)} & \xrightarrow{H_1} & (\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(i)})^* \\ & \searrow H_0 & \downarrow H_2 \\ & & \lambda_{\Phi_2^\dagger\Phi_1^\dagger\mathcal{O}^\dagger}^{(i)} \end{array}$$

in which H_2 has already been solved in (26). Since both H_0 and H_2 are linear transformations, $H_1 = H_0 \cdot H_2^{-1}$ is linear as well. In practice, we will firstly calculate H_1 based on the unitarity of superconformal partial waves and then solve H_0 in terms of H_1 and H_2 .

The transformation H_1 has been solved in Appendix C, and the most general $\mathcal{N} = 1$ superconformal block $\mathcal{G}_{\Delta,\ell}^{r,\bar{r}}$ is written in terms of $\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(i)}$ and $(\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(i)})^*$. Transformation

from $(\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(i)})^*$ to $\lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(i)}$ has been solved in (26), its inverse map gives $H_2(\tilde{r})$:

$$\begin{pmatrix} (\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(0)})^* \\ (\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(2)})^* \\ (\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(1)})^* \\ (\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(3)})^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\tilde{r}^2 & 1 & \tilde{r} & \frac{1}{2}\tilde{r} \\ \tilde{r} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(0)} \\ \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(2)} \\ \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(1)} \\ \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(3)} \end{pmatrix}, \quad (67)$$

and it satisfies

$$H_2(r) \cdot H_2(-r) = I_{4 \times 4}, \quad (68)$$

which is expected since the coefficients are invariant by taking complex conjugate twice.

It is straightforward to get transformation H_0 by combining the results of H_1 and H_2 . Here we do not present the explicit expression of H_0 . The $\mathcal{N} = 1$ superconformal block is

$$\mathcal{G}_{\Delta, \ell}^{r, \tilde{r}} = a_1 g_{\Delta, \ell}^{r, \tilde{r}} + a_2 g_{\Delta+1, \ell+1}^{r, \tilde{r}} + a_3 g_{\Delta+1, \ell-1}^{r, \tilde{r}} + a_4 g_{\Delta+2, \ell}^{r, \tilde{r}}, \quad (69)$$

in which the coefficients of individual conformal blocks a_i are written in terms of $\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)}$ and $\lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(i)}$

$$a_1 = \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(0)}, \quad (70)$$

$$a_2 = \frac{\Delta + \ell}{(\Delta + \ell + 1)(\Delta - R + \ell)(\Delta + R + \ell)} \left(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)} + \frac{r(\Delta - R + \ell)}{2(\Delta + \ell)} \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)} \right) \\ \times \left(\lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(1)} + \frac{\tilde{r}(\Delta + R + \ell)}{2(\Delta + \ell)} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(0)} \right), \quad (71)$$

$$a_3 = \frac{\ell + 2 - \Delta}{(-\Delta + \ell + 1)(-\Delta - R + \ell + 2)(-\Delta + R + \ell + 2)} \\ \times \left(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)} + \frac{\ell + 1}{\ell} \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)} + \frac{r(-\Delta + R + \ell + 2)}{2(-\Delta + \ell + 2)} \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)} \right) \\ \times \left(\lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(1)} + \frac{\ell + 1}{\ell} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(3)} + \frac{\tilde{r}(-\Delta - R + \ell + 2)}{2(-\Delta + \ell + 2)} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(0)} \right), \quad (72)$$

$$a_4 = \frac{4(\Delta - 1)^2(-\Delta + \ell + 2)(\Delta + \ell)}{\Delta^2(\ell + 1 - \Delta)(\Delta + \ell + 1)(\ell + 2 - R - \Delta)(\ell + 2 + R - \Delta)(\Delta - R + \ell)(\Delta + R + \ell)} \times \\ \left[-\frac{(\Delta - R + \ell)(R(\ell(\ell + 2) - \Delta(\Delta + r^2 - 2)) + (\ell + 2 - \Delta)((\Delta + \ell)^2 - \Delta r^2))}{8(\Delta - 1)(\ell + 2 - \Delta)(\Delta + \ell)} \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)} \right. \\ \left. + \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(2)} + \frac{r(\Delta(R + 2 - \Delta) + \ell(\ell + 2))}{2(\ell + 2 - \Delta)(\Delta + \ell)} \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)} + \frac{r(\Delta - R + \ell)}{4(\Delta + \ell)} \lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)} \right] \times \\ \left[\frac{(\Delta + R + \ell)(R(\ell(\ell + 2) - \Delta(\Delta + \tilde{r}^2 - 2)) - (\ell + 2 - \Delta)((\Delta + \ell)^2 - \Delta \tilde{r}^2))}{8(\Delta - 1)(\ell + 2 - \Delta)(\Delta + \ell)} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(0)} \right. \\ \left. + \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(2)} + \frac{\tilde{r}(\Delta(-R + 2 - \Delta) + \ell(\ell + 2))}{2(\ell + 2 - \Delta)(\Delta + \ell)} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(1)} + \frac{\tilde{r}(\Delta + R + \ell)}{4(\Delta + \ell)} \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(3)} \right]. \quad (73)$$

Comparing with the superconformal blocks (C23-C26) in terms of $(\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(i)})^*$, above superconformal blocks show improved symmetry that terms appear in pairs with correspondences

$$\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)} \leftrightarrow \lambda_{\Phi_3 \Phi_4 \mathcal{O}^\dagger}^{(i)}, \quad r \leftrightarrow \tilde{r}, \quad R \leftrightarrow -R. \quad (74)$$

Taking $r = \tilde{r} = R = 0$, the coefficients a_i presented in (70-73) reduce to the results obtained in [32]. For non-vanishing r, \tilde{r}, R , if certain fields Φ 's in four-point function satisfy shortening conditions, like chirality, the tensor structures can be simplified and there will be strong constraints on the coefficients $\lambda_{\Phi_i \Phi_j \mathcal{O}}^{(i)}$. In this case the superconformal blocks can be conveniently solved through superconformal Casimir approach [17, 22, 31]. As a non-trivial check, we compare our work with previous results on $\mathcal{N} = 1$ superconformal blocks obtained from superconformal Casimir approach [17, 22].

In [17] superconformal blocks in SCFTs with four supercharges have been studied. The authors considered four-point function $\langle \Phi_1(1)X_1(2, \bar{2})\Phi_2(3)X_2(4, \bar{4}) \rangle$, in which $\Phi_{1,2}$ are chiral, while $X_{1,2}$ are scalars with arbitrary superconformal weights. As shown in (28), chirality conditions of Φ_1 and Φ_2 lead to following constraints on the coefficients

$$(\lambda_{\Phi_1 X_1 \mathcal{O}}^{(0)}, \lambda_{\Phi_1 X_1 \mathcal{O}}^{(2)}, \lambda_{\Phi_1 X_1 \mathcal{O}}^{(1)}, \lambda_{\Phi_1 X_1 \mathcal{O}}^{(3)}) = \lambda_{\Phi_1 X_1 \mathcal{O}}(1, e_1(2e_1 - \ell), -2e_1, \ell), \quad (75)$$

$$(\lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}^{(0)}, \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}^{(2)}, \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}^{(1)}, \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}^{(3)}) = \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}(1, e_2(2e_2 - \ell), -2e_2, \ell), \quad (76)$$

where parameters e_1 and e_2 are

$$e_1 = \frac{1}{4}(\Delta + \ell + 2r + R), \quad e_2 = \frac{1}{4}(2 - \Delta + \ell + 2\tilde{r} - R), \quad (77)$$

and here the scaling dimension differences r and \tilde{r} become $r = \Delta_{\Phi_1} - \Delta_{X_1}$, $\tilde{r} = \Delta_{\Phi_2} - \Delta_{X_2}$.

Plugging these constraints in (70-73), coefficients of conformal blocks in $\mathcal{G}_{\Delta, \ell}^{r, \tilde{r}}$ turn into

$$a_1 = \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \quad (78)$$

$$a_2 = \frac{(\Delta + r + \ell)(\Delta + \tilde{r} + \ell)}{4(\Delta + \ell)(\Delta + \ell + 1)} \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \quad (79)$$

$$a_3 = \frac{(\Delta + r - \ell - 2)(\Delta + \tilde{r} - \ell - 2)}{4(-\Delta + \ell + 1)(-\Delta + \ell + 2)} \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \quad (80)$$

$$a_4 = \frac{(\Delta + r - \ell - 2)(\Delta + \tilde{r} - \ell - 2)(\Delta + r + \ell)(\Delta + \tilde{r} + \ell)}{16(-\Delta + \ell + 1)(-\Delta + \ell + 2)(\Delta + \ell)(\Delta + \ell + 1)} \lambda_{\Phi_1 X_1 \mathcal{O}} \lambda_{\Phi_2 X_2 \mathcal{O}^\dagger}, \quad (81)$$

which are in agreement with the results obtained in [17]. $\mathcal{N} = 1, 2$ superconformal blocks are also presented in [22], in which the four-point function consists of chiral-antichiral scalars with arbitrary $U(1)$ R-charges. For $\mathcal{N} = 1$ case, the superconformal blocks are similar to above expressions and are well consistent with our results.

VI. DISCUSSION

In this work we have computed the most general $\mathcal{N} = 1$ superconformal partial waves $\mathcal{W}_{\mathcal{O}} \propto \langle \Phi_1 \Phi_2 | \mathcal{O} | \Phi_3 \Phi_4 \rangle$, in which the scalars Φ_i have arbitrary scaling dimensions and $U(1)$ R-charges. Our computations are based on the superembedding space formalism and supershadow approach, which provide a systematic way to study $\mathcal{N} = 1$ superconformal blocks. Unitarity of SCFTs has been used to evaluate the coefficients in the three-point function of supershadow operator. Besides, it shows deep connections between conformal field theories and mathematical properties of hypergeometric functions throughout the computations. Our results nicely reproduce all the known results on the $\mathcal{N} = 1$ superconformal blocks under certain restrictions.

The superconformal blocks of operators with arbitrary scaling dimensions and R-charges are crucial ingredients for the mixed operator conformal bootstrap, and our results provide necessary materials for bootstrapping any $\mathcal{N} = 1$ SCFTs. An attractive problem is the $4D$ $\mathcal{N} = 1$ minimal SCFT, which has no Lagrangian description and its existence is only revealed in superconformal bootstrap [7, 53]. More details of the theory are expected to be studied through bootstrapping the mixed operator correlators [54]. Our current results on the SCFTs are limited to $4D$ $\mathcal{N} = 1$ scalars, and obviously it can be generalized from three aspects: dimension of spacetime, number of supercharges and spin of the fields in four-point correlator. The supershadow approach has impressive successes in solving $4D$ $\mathcal{N} = 1$ scalar superconformal blocks, we hope this method, and its generalizations can be used to obtain the superconformal blocks of spinning operators in other dimensional spacetime with different supercharges.

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Appendix A: Gegenbauer Polynomial and Some Identities

It has been shown in [31, 32, 39, 57] that \mathcal{N}_ℓ appearing in the superconformal/conformal partial wave integration directly relates to Gegenbauer polynomial $C_\ell^{(\lambda)}(x)$

$$\mathcal{N}_\ell \equiv (\bar{S}1\bar{2}S)^\ell \overleftrightarrow{\mathcal{D}}_\ell (\bar{T}3\bar{4}T)^\ell = \frac{1}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^\ell = (-1)^\ell y^{\frac{\ell}{2}} C_\ell^{(1)}(x), \quad (\text{A1})$$

in which

$$x \equiv \frac{\langle \bar{2}1\bar{0}3\bar{4}0 \rangle}{2\sqrt{y}}, \quad y \equiv \frac{1}{2^6} \langle \bar{0}1 \rangle \langle \bar{2}0 \rangle \langle \bar{0}3 \rangle \langle \bar{4}0 \rangle \langle \bar{2}1 \rangle \langle \bar{4}3 \rangle. \quad (\text{A2})$$

Giving $\theta_{\text{ext}} = 0$, variables x and y turn into

$$x \longrightarrow x_0 \equiv -\frac{X_{13}X_{20}X_{40}}{2\sqrt{X_{10}X_{20}X_{30}X_{40}X_{12}X_{34}}} - (1 \leftrightarrow 2) - (3 \leftrightarrow 4), \quad (\text{A3})$$

$$y \longrightarrow y_0 \equiv \frac{1}{2^{12}} X_{10}X_{20}X_{30}X_{40}X_{12}X_{34}, \quad (\text{A4})$$

in which the supertraces $\langle i\bar{j} \rangle$ have been reduced to inner products of six dimensional vectors X_{ij} . Besides we follow the conventions used in [32] that the super-parameters are replaced by

$$\mathcal{S} \rightarrow S, \quad \bar{\mathcal{S}} \rightarrow \bar{S}, \quad \mathcal{N}_\ell \rightarrow N_\ell \quad (\text{A5})$$

after setting $\theta_{\text{ext}} = 0$, and the Gegenbauer polynomial N_ℓ reads

$$N_\ell = (\bar{S}1\bar{2}S)^\ell \overleftrightarrow{\mathcal{D}}_\ell (\bar{T}3\bar{4}T)^\ell = \frac{1}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^\ell. \quad (\text{A6})$$

Giving $0 = \bar{0}$, one can show

$$S\bar{2}1\bar{0}3\bar{4}T = \frac{1}{4}X_{10}S\bar{2}3\bar{4}T - \frac{1}{4}X_{20}S\bar{1}3\bar{4}T = \frac{1}{4}X_{30}S\bar{2}1\bar{4}T - \frac{1}{4}X_{40}S\bar{2}1\bar{3}T \quad (\text{A7})$$

based on the Clifford algebra and the transverse conditions of auxiliary fields $S\bar{0} = \bar{0}T = 0$. It clearly shows that $S\bar{2}1\bar{0}3\bar{4}T$ is antisymmetric under $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$.

Let us consider following formulas related to the Gegenbauer polynomials

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{10}S\bar{2}3\bar{4}T + X_{20}S\bar{1}3\bar{4}T + X_{10}S\bar{2}4\bar{3}T + X_{20}S\bar{1}4\bar{3}T), \quad (\text{A8})$$

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{10}S\bar{2}3\bar{4}T + X_{20}S\bar{1}3\bar{4}T - X_{10}S\bar{2}4\bar{3}T - X_{20}S\bar{1}4\bar{3}T), \quad (\text{A9})$$

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{10}S\bar{2}3\bar{4}T - X_{20}S\bar{1}3\bar{4}T + X_{10}S\bar{2}4\bar{3}T - X_{20}S\bar{1}4\bar{3}T), \quad (\text{A10})$$

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{10}S\bar{2}3\bar{4}T - X_{20}S\bar{1}3\bar{4}T - X_{10}S\bar{2}4\bar{3}T + X_{20}S\bar{1}4\bar{3}T), \quad (\text{A11})$$

which are symmetric or anti-symmetric under coordinate interchanges $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$. These polynomials appear in the conformal integral (18) from differentiations $(\partial_{\bar{0}}z) \cdot (\partial_{\bar{0}}N_\ell)$ or $(\partial_{\bar{0}}\frac{1}{D_\ell}) \cdot (\partial_{\bar{0}}N_\ell)$ and inherit the symmetry properties from tensor structure terms in (45). We need to find their close relationships with Gegenbauer polynomials to accomplish the conformal integration (18).

Formulas in (A8) and (A11) are invariant under simultaneous coordinate interchange $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, and they can be easily simplified into compact form N_ℓ . Specifically, the formula (A11) gives

$$8(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^\ell \propto N_\ell, \quad (\text{A12})$$

while for (A8), one can show that it reduces to

$$\begin{aligned} & \frac{1}{4}(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{10}X_{34}S\bar{2}T + X_{20}X_{34}S\bar{1}T) \\ &= \frac{\ell(\ell+1)}{8} X_{10}X_{20}X_{34} (\partial_S 0 \partial_T)^{\ell-1} (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \\ &\propto X_{10}X_{20}X_{34}N_{\ell-1}. \end{aligned} \quad (\text{A13})$$

In contrast, formulas in (A9) and (A10) are antisymmetric under $1 \leftrightarrow 2$, $3 \leftrightarrow 4$. It is easy to show that formula (A10) vanishes.

Similarly, we can reduce following formulas to compact forms proportional to N_ℓ :

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{30}S\bar{2}1\bar{4}T + X_{40}S\bar{2}1\bar{3}T + X_{30}S\bar{1}2\bar{4}T + X_{40}S\bar{1}2\bar{3}T), \quad (\text{A14})$$

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{30}S\bar{2}1\bar{4}T + X_{40}S\bar{2}1\bar{3}T - X_{30}S\bar{1}2\bar{4}T - X_{40}S\bar{1}2\bar{3}T), \quad (\text{A15})$$

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{30}S\bar{2}1\bar{4}T - X_{40}S\bar{2}1\bar{3}T + X_{30}S\bar{1}2\bar{4}T - X_{40}S\bar{1}2\bar{3}T), \quad (\text{A16})$$

$$(\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} (X_{30}S\bar{2}1\bar{4}T - X_{40}S\bar{2}1\bar{3}T - X_{30}S\bar{1}2\bar{4}T + X_{40}S\bar{1}2\bar{3}T), \quad (\text{A17})$$

except (A15).

The formulas (A9) and (A15) can not be simply written in terms of N_ℓ , nevertheless, their relationships with the Gegenbauer polynomials are given in the recursion equations, which can be used to obtain the final results of conformal integrations they involve in.

Denote

$$R_\ell \equiv \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \times \\ (X_{10}S\bar{2}3\bar{4}T + X_{20}S\bar{1}3\bar{4}T - X_{10}S\bar{2}4\bar{3}T - X_{20}S\bar{1}4\bar{3}T), \quad (\text{A18})$$

$$P_\ell \equiv \frac{\ell}{\ell!^2} (\partial_S 0 \partial_T)^\ell (S\bar{2}1\bar{0}3\bar{4}T)^{\ell-1} \times \\ (X_{30}S\bar{2}1\bar{4}T + X_{40}S\bar{2}1\bar{3}T - X_{30}S\bar{1}2\bar{4}T - X_{40}S\bar{1}2\bar{3}T), \quad (\text{A19})$$

and

$$\Delta_A \equiv \frac{1}{8} (X_{20}X_{40}X_{13} - X_{20}X_{30}X_{14} + X_{10}X_{40}X_{23} - X_{10}X_{30}X_{24}), \quad (\text{A20})$$

$$\Delta_B \equiv \frac{1}{8} (X_{20}X_{40}X_{13} + X_{20}X_{30}X_{14} - X_{10}X_{40}X_{23} - X_{10}X_{30}X_{24}). \quad (\text{A21})$$

Note the sign differences among x_0 , Δ_A and Δ_B . The crucial properties of R_ℓ and P_ℓ are that they satisfy the following mutual recursion relations:

$$R_\ell = \ell \Delta_A N_{\ell-1} + \frac{1}{2^6} X_{10} X_{20} X_{34} P_{\ell-1}, \quad (\text{A22})$$

$$P_\ell = \ell \Delta_B N_{\ell-1} + \frac{1}{2^6} X_{30} X_{40} X_{12} R_{\ell-1}, \quad (\text{A23})$$

which leads to the independent recursion relations of R_ℓ and P_ℓ :

$$R_\ell = \ell \Delta_A N_{\ell-1} + \frac{1}{2^6} (\ell-1) X_{10} X_{20} X_{34} \Delta_B N_{\ell-2} + y_0 R_{\ell-2}, \quad (\text{A24})$$

$$P_\ell = \ell \Delta_B N_{\ell-1} + \frac{1}{2^6} (\ell-1) X_{30} X_{40} X_{12} \Delta_A N_{\ell-2} + y_0 P_{\ell-2}. \quad (\text{A25})$$

Above two recursion equations are needed to determine the conformal integrations of the antisymmetric terms in (46).

Appendix B: Superconformal Integrations of Symmetric Terms

The superconformal partial waves $\mathcal{W}_\mathcal{O}$ are largely determined by the tensor structures in (45). These terms are separated into two parts: invariant and antisymmetric terms according to their transformations under coordinate interchange $1 \leftrightarrow 2$, $3 \leftrightarrow 4$. Here we show the main steps toward contributions of invariant terms on $\mathcal{W}_\mathcal{O}$. Due to the gauge adopted in (21), we can obtain the results straightforwardly, similar to the steps used in [32] but generalized to Φ_i 's with arbitrary superconformal weights.

As discussed before, there are two steps to accomplish the superconformal integrations for $\mathcal{W}_\mathcal{O}$: partial derivatives and conformal integration. The partial derivatives can be obtained by the same steps provided in [32] with coefficients replacements

$$\frac{\ell + \Delta}{2} \rightarrow 2\delta, \quad \frac{2 + \ell - \Delta}{2} \rightarrow 2\delta'. \quad (\text{B1})$$

The conformal integrations are modified accordingly, specifically there are new terms proportional to the scaling dimension differences r, \tilde{r} :

$$\int D^4 X_0 \frac{X_{12}}{X_{10} X_{20}} \frac{N_\ell}{D_\ell} \Big|_{\bar{0}=0} = \frac{c_\ell \xi_{\Delta+2,2-\Delta,\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} g_{\Delta+2,\ell}^{r,\tilde{r}}(u,v), \quad (\text{B2})$$

$$\int D^4 X_0 \frac{X_{34}}{X_{30} X_{40}} \frac{N_\ell}{D_\ell} \Big|_{\bar{0}=0} = \frac{c_\ell \xi_{\Delta,4-\Delta,\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} g_{\Delta,\ell}^{r,\tilde{r}}(u,v), \quad (\text{B3})$$

$$\int D^4 X_0 X_{12} X_{34} \frac{N_{\ell-1}}{D_\ell} \Big|_{\bar{0}=0} = \frac{c_{\ell-1} \xi_{\Delta+1,3-\Delta,\tilde{r},\ell-1}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} g_{\Delta+1,\ell-1}^{r,\tilde{r}}(u,v), \quad (\text{B4})$$

$$\begin{aligned} \int D^4 X_0 \left[\frac{X_{13}}{X_{10} X_{30}} + \frac{X_{23}}{X_{20} X_{30}} + \frac{X_{14}}{X_{10} X_{40}} + \frac{X_{24}}{X_{20} X_{40}} \right] \frac{N_\ell}{D_\ell} \Big|_{\bar{0}=0} = \\ \frac{c_\ell \xi_{\Delta+1,3-\Delta,1+\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} \left[\frac{4(\tilde{r}^2 + (\Delta - \ell - 2)(\Delta + \ell))}{(\tilde{r} + \Delta - \ell - 2)(\tilde{r} + \Delta + \ell)} g_{\Delta,\ell}^{r,\tilde{r}} \right. \\ + \frac{(r^2 + (\Delta - \ell - 2)(\Delta + \ell))(\tilde{r} - \Delta - \ell)(\tilde{r} - \Delta + \ell + 2)}{4(\Delta - \ell - 2)(\Delta - \ell - 1)(\Delta + \ell)(\Delta + \ell + 1)} g_{\Delta+2,\ell}^{r,\tilde{r}} \\ + \frac{r\tilde{r}(\tilde{r} - \Delta - \ell)}{(\Delta + \ell)(\Delta + \ell + 1)(\tilde{r} + \Delta - \ell - 2)} g_{\Delta+1,\ell+1}^{r,\tilde{r}} \\ \left. + \frac{r\tilde{r}(\tilde{r} - \Delta + \ell + 2)}{(\Delta - \ell - 2)(\Delta - \ell - 1)(\tilde{r} + \Delta + \ell)} g_{\Delta+1,\ell-1}^{r,\tilde{r}} \right], \quad (\text{B5}) \end{aligned}$$

$$\begin{aligned} \int D^4 X_0 \left[\frac{X_{13}}{X_{10} X_{30}} - \frac{X_{23}}{X_{20} X_{30}} - \frac{X_{14}}{X_{10} X_{40}} + \frac{X_{24}}{X_{20} X_{40}} \right] \frac{N_\ell}{D_\ell} \Big|_{\bar{0}=0} = \\ \frac{c_\ell \xi_{\Delta+1,3-\Delta,1+\tilde{r},\ell}}{X_{12}^{\frac{1}{2}(\Delta-\ell)} X_{34}^{-\frac{1}{2}(\Delta+\ell-2)}} \left(\frac{X_{24}}{X_{14}} \right)^{\frac{r}{2}} \left(\frac{X_{14}}{X_{13}} \right)^{\frac{\tilde{r}}{2}} \left[\frac{(\Delta - \ell - 2)(-\tilde{r} + \Delta + \ell)}{(\Delta + \ell + 1)(\tilde{r} + \Delta - \ell - 2)} g_{\Delta+1,\ell+1}^{r,\tilde{r}} \right. \\ \left. + \frac{(\Delta + \ell)(-\tilde{r} + \Delta - \ell - 2)}{(\Delta - \ell - 1)(\tilde{r} + \Delta + \ell)} g_{\Delta+1,\ell-1}^{r,\tilde{r}} \right]. \quad (\text{B6}) \end{aligned}$$

Appendix C: Solution of the Shadow Coefficients Transformation

Here we solve the linear transformation H_1 between the supershadow coefficients $\lambda_{\Phi_2^\dagger \Phi_1^\dagger \bar{\mathcal{O}}}^{(i)}$ and $(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)})^*$. As proposed in [32], the unitarity of superconformal partial wave plays a crucial role in determining H_1 .

The linear transformation H_1 is described by a 4×4 matrix

$$\begin{pmatrix} \lambda_{\Phi_2^\dagger \Phi_1^\dagger \tilde{\mathcal{O}}}^{(0)} \\ \lambda_{\Phi_2^\dagger \Phi_1^\dagger \tilde{\mathcal{O}}}^{(2)} \\ \lambda_{\Phi_2^\dagger \Phi_1^\dagger \tilde{\mathcal{O}}}^{(1)} \\ \lambda_{\Phi_2^\dagger \Phi_1^\dagger \tilde{\mathcal{O}}}^{(3)} \end{pmatrix} = \begin{pmatrix} a & b & e & g \\ c & d & f & h \\ u & v & p & k \\ w & t & q & s \end{pmatrix} \begin{pmatrix} (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(0)})^* \\ (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(2)})^* \\ (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(1)})^* \\ (\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(3)})^* \end{pmatrix}. \quad (\text{C1})$$

Note that in [32] the 4×4 matrix is block diagonal protected by the parity of coefficients under coordinate exchange in the three-point function. While for the three-point function with general superconformal weights, the coordinate exchange symmetry is broken by arbitrary superconformal weights, therefore in our case the 4×4 matrix is not simply block diagonal, nevertheless the unitarity, together with extra constraint is still useful to solve the transformation H_1 .

Giving $\Phi_3 = \Phi_2^\dagger$ and $\Phi_4 = \Phi_1^\dagger$, unitarity requires that the four coefficients a_i of conformal blocks appearing in the superconformal blocks $\mathcal{G}_{\Delta, \ell}^{r, \tilde{r}}$ are positive. By transforming coefficients $\lambda_{\Phi_2^\dagger \Phi_1^\dagger \tilde{\mathcal{O}}}^{(i)}$ to $(\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)})^*$, this is equivalent to the following equations:

$$\left(-\delta' \left[\frac{(2-\Delta)\tilde{r}^2 - (\ell+2-\Delta)(\Delta+\ell)}{(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)} 2\delta + 1 \right], 1, \frac{\tilde{r}((\Delta-2)R + (-\Delta+\ell+2)(\Delta+\ell))}{2(-\Delta+\ell+2)(\Delta+\ell)}, \frac{\tilde{r}(R+\ell+2-\Delta)}{4(-\Delta+\ell+2)} \right) \cdot H_1 \propto (1, 0, 0, 0), \quad (\text{C2})$$

$$\left(\frac{\tilde{r}(R+\ell+2-\Delta)}{2(-\Delta+\ell+2)}, 0, 1, 0 \right) \cdot H_1 \propto \left(\frac{r(\Delta-R+\ell)}{2(\Delta+\ell)}, 0, 1, 0 \right), \quad (\text{C3})$$

$$\left(\frac{\tilde{r}(\Delta-R+\ell)}{2(\Delta+\ell)}, 0, 1, \frac{\ell+1}{\ell} \right) \cdot H_1 \propto \left(\frac{r(-\Delta+R+\ell+2)}{2(-\Delta+\ell+2)}, 0, 1, \frac{\ell+1}{\ell} \right), \quad (\text{C4})$$

$$\begin{aligned} (1, 0, 0, 0) \cdot H_1 &\propto \left(-\delta \left(1 - 2\delta' \frac{r^2\Delta - (\Delta+\ell)(-\Delta+\ell+2)}{(\Delta-1)(\ell+2-\Delta)(\Delta+\ell)} \right), \right. \\ &\quad \left. 1, \frac{r(\Delta(-\Delta+R+2) + \ell(\ell+2))}{2(-\Delta+\ell+2)(\Delta+\ell)}, \frac{r(\Delta-R+\ell)}{4(\Delta+\ell)} \right), \end{aligned} \quad (\text{C5})$$

in which $\tilde{r} = -r$. From above equation groups we can solve 15 out of 16 H_1 's elements (except c) up to three re-scaling coefficients.

Then we consider two three-point functions $\langle \Phi X \mathcal{O} \rangle$ and $\langle X \Phi^\dagger \tilde{\mathcal{O}} \rangle$, in which $\Phi : (0, 0, q_1, 0)$ is a chiral field while $X : (0, 0, q_2, q_2)$ is real ². Such kind of three-point

² X could be any scalar and the results will be the same, here we set X as real for convenience.

function has been studied in (28). Due to the chirality of Φ , the four coefficients actually satisfy the constraint

$$(\lambda_{\Phi X \mathcal{O}}^{(0)}, \lambda_{\Phi X \mathcal{O}}^{(2)}, \lambda_{\Phi X \mathcal{O}}^{(1)}, \lambda_{\Phi X \mathcal{O}}^{(3)}) = \lambda_{\Phi X \mathcal{O}}(1, \delta(2\delta - \ell), -2\delta, \ell), \quad (\text{C6})$$

$$(\lambda_{X\Phi^\dagger \bar{\mathcal{O}}}^{(0)}, \lambda_{X\Phi^\dagger \bar{\mathcal{O}}}^{(2)}, \lambda_{X\Phi^\dagger \bar{\mathcal{O}}}^{(1)}, \lambda_{X\Phi^\dagger \bar{\mathcal{O}}}^{(3)}) = \lambda_{X\Phi^\dagger \bar{\mathcal{O}}}(1, \delta'(2\delta' - \ell), -2\delta', \ell), \quad (\text{C7})$$

in which $\delta = \frac{\Delta + \ell + R + 2r}{4}$ and $\delta' = \frac{2 - \Delta + \ell + R}{4}$ with $R = -q_1$, $r = q_1 - 2q_2$. then the transformation between coefficients in (C7) and the complex conjugate of (C6) gives

$$\begin{pmatrix} 1 \\ \delta'(2\delta' - \ell) \\ -2\delta' \\ \ell \end{pmatrix} \propto \begin{pmatrix} a & b & e & g \\ c & d & f & h \\ u & v & p & k \\ w & t & q & s \end{pmatrix} \begin{pmatrix} 1 \\ \delta(2\delta - \ell) \\ -2\delta \\ \ell \end{pmatrix}. \quad (\text{C8})$$

Plugging the solutions of equation groups (C2-C5) into (C8), we can solve all the 16 elements in H_1 and three re-scaling coefficient up to the re-scaling coefficient of (C8), denoted as z_* .

The results are

$$\alpha_* \times \begin{pmatrix} a_* & -\frac{8(\ell - \Delta + 2)(\ell + \Delta)}{\ell + \Delta - R} & -\frac{4r(\ell(\ell + 2) + (R - \Delta + 2)\Delta)}{\ell + \Delta - R} & -2r(\ell - \Delta + 2) \\ c_* & d_* & f_* & h_* \\ u_* & -\frac{4r(R + \ell - \Delta + 2)(\ell + \Delta)}{\ell + \Delta - R} & p_* & -r^2(R + \ell - \Delta + 2) \\ w_* & \frac{8rR\ell}{\ell + \Delta - R} & \frac{4\ell(Rr^2 + (\ell - \Delta + 2)\Delta(\ell + \Delta))}{\ell + \Delta - R} & s_* \end{pmatrix} \quad (\text{C9})$$

in which the elements with long expressions are abbreviated as

$$\alpha_* = \frac{z_*(\Delta - 1)(\Delta - R + \ell)}{\Delta(-\Delta - r + \ell + 2)(\Delta + r + \ell)(-\Delta - R + \ell + 2)(\Delta + R + \ell)}, \quad (\text{C10})$$

$$a_* = \frac{R(\ell(\ell + 2) - \Delta(\Delta + r^2 - 2)) + (-\Delta + \ell + 2)((\Delta + \ell)^2 - \Delta r^2)}{\Delta - 1}, \quad (\text{C11})$$

$$d_* = \frac{R + \ell + 2 - \Delta}{(\Delta - 1)(R - \ell - \Delta)} (\Delta^2(r^2 - R + \ell + 4) - \Delta^3 + \Delta((2 - r^2)R + \ell(r^2 + \ell) - 4) + \ell(\ell + 2)(R - \ell - 2)), \quad (\text{C12})$$

$$h_* = \frac{r(R + \ell + 2 - \Delta)}{4(\Delta - 1)} (-\Delta(\Delta^2 + \Delta - 2r^2 - 4) - (\Delta - 1)R(\Delta + \ell + 2) - (\Delta + 1)\ell^2 - 2((\Delta - 1)\Delta + 2)\ell - 4), \quad (\text{C13})$$

$$u_* = -\frac{r(R + \ell + 2 - \Delta)}{2(-1 + \Delta)} (-\Delta((\Delta - 3)\Delta - 2r^2 + 4) + (\Delta - 1)R(\ell - \Delta) - (\Delta + 1)\ell^2 + 2(\Delta - 3)\Delta\ell), \quad (\text{C14})$$

$$p_* = \frac{R + \ell + 2 - \Delta}{(\Delta - 1)(R - \ell - \Delta)} (r^2(\Delta(3\Delta - R - 2) + (3\Delta - 2)\ell) + \Delta(-\Delta + \ell + 2)(\Delta + \ell)(\Delta + R - \ell - 2)), \quad (\text{C15})$$

$$s_* = \frac{1}{\Delta - 1} (r^2((\Delta - 2)R + \Delta(\Delta - \ell - 2)) + \Delta(-\Delta + \ell + 2)(\Delta + \ell)(\Delta + R + \ell)), \quad (\text{C16})$$

$$w_* = r\ell(4\Delta + (R + \ell - \Delta)(R + 2\Delta)), \quad (\text{C17})$$

and

$$c_* = \frac{\Delta + R - \ell - 2}{8(\Delta - 1)(\ell + 2 - \Delta - R)(\Delta - R + \ell)} (4(\Delta - 1)\Delta r^2 R^3 + (\Delta - 1)R^4(\ell + 2 - \Delta)(\Delta + \ell) - 4(\Delta - 1)\Delta r^2 R \\ ((\Delta - 4)\Delta - 2r^2 + 3\ell(\ell + 2) + 6) + 2R^2(\ell(\ell + 2)(\ell(\ell + 2) + 2) + \Delta^5 - 5\Delta^4 - 2\Delta^3(r^2 - 5) + 2\Delta^2(r^2 - 5) \\ - \Delta(2r^4 - 2r^2(\ell + 1)^2 + \ell(\ell + 2)(\ell(\ell + 2) + 2) - 4) + (\ell + 2 - \Delta)(\Delta + \ell)(\Delta^5 - 5\Delta^4 - 2\Delta^3(2r^2 + \ell(\ell + 2) \\ - 4) + 2\Delta^2(6r^2 + 3\ell(\ell + 2) - 2) + \Delta(4r^4 - 4r^2(\ell(\ell + 2) + 3) + \ell^3(\ell + 4) - 8\ell) - \ell^2(\ell + 2)^2)), \quad (C18)$$

$$f_* = \frac{R + \ell + 2 - \Delta}{2(\Delta - 1)(\Delta - R + \ell)} (\ell(\ell + 2)(-R + \ell + 2) + \Delta(R(2r^2 - \ell(\ell + 3) - 4) + \ell(-2r^2 + \ell^2 + \ell + 4) + R^2 + 4) \\ + \Delta^2(-2r^2 - R^2 + R(\ell + 2) + (\ell - 3)\ell) + \Delta^3(\ell - 3) + \Delta^4). \quad (C19)$$

The transformation H_1 presented above seems to be rather cumbersome, however it does satisfy following simple relation

$$H_1(\Delta, R, r) \cdot H_1(\Delta \rightarrow 2 - \Delta, R \rightarrow -R, r \rightarrow -r) \propto I_{4 \times 4}, \quad (C20)$$

which is expected since by applying the supershadow transformation twice we go back to the original coefficients. Setting the Eq. (C20) to be strictly equal, the overall coefficient z_* can be fixed up to a factor z_x satisfying

$$z_x(\Delta, R, r) \cdot z_x(\Delta \rightarrow 2 - \Delta, R \rightarrow -R, r \rightarrow -r) = 1, \quad (C21)$$

which, however, has no effect on the superconformal block functions.

Besides the three-point correlators $\langle \Phi X \mathcal{O} \rangle$ and $\langle X \Phi^\dagger \mathcal{O} \rangle$, we can also partially fix the coefficients in the three-point correlators like $\langle \Phi^\dagger X \mathcal{O} \rangle$, $\langle X \Phi \mathcal{O} \rangle$ and their supershadow duals. Their coefficients are expected to be related to the shadow coefficients by H_1 with proper redefinitions of parameter r and R . One can show that indeed above solution of H_1 can realize the transformation of shadow coefficients with parameters $R \rightarrow -R$ and $r \rightarrow -r$, respectively.

Under transformation H_1 , the coefficients $\lambda_{\Phi_3 \Phi_4 \mathcal{O}}^{(i)}$ in (61-64) can be mapped to $(\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(i)})^*$, and now we are ready to write down the most general $\mathcal{N} = 1$ superconformal block $\mathcal{G}_{\Delta, \ell}^{r, \tilde{r}}$ in terms of three-point coefficients $\lambda_{\Phi_1 \Phi_2 \mathcal{O}}^{(i)}$ and $(\lambda_{\Phi_4^\dagger \Phi_3^\dagger \mathcal{O}}^{(i)})^*$:

$$\mathcal{G}_{\Delta, \ell}^{r, \tilde{r}} = a_1 g_{\Delta, \ell}^{r, \tilde{r}} + a_2 g_{\Delta+1, \ell+1}^{r, \tilde{r}} + a_3 g_{\Delta+1, \ell-1}^{r, \tilde{r}} + a_4 g_{\Delta+2, \ell}^{r, \tilde{r}}, \quad (C22)$$

where the coefficients of individual conformal blocks a_i are

$$a_1 = \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)}(\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(0)})^*, \quad (\text{C23})$$

$$a_2 = \frac{\Delta + \ell}{(\Delta + \ell + 1)(\Delta - R + \ell)(\Delta + R + \ell)} \left(\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} + \frac{r(\Delta - R + \ell)}{2(\Delta + \ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} \right) \\ \times \left((\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(1)})^* - \frac{\tilde{r}(\Delta - R + \ell)}{2(\Delta + \ell)} (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(0)})^* \right), \quad (\text{C24})$$

$$a_3 = \frac{\ell + 2 - \Delta}{(-\Delta + \ell + 1)(-\Delta - R + \ell + 2)(-\Delta + R + \ell + 2)} \\ \times \left(\lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} + \frac{\ell + 1}{\ell} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(3)} + \frac{r(-\Delta + R + \ell + 2)}{2(-\Delta + \ell + 2)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} \right) \\ \times \left((\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(1)})^* + \frac{\ell + 1}{\ell} (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(3)})^* - \frac{\tilde{r}(-\Delta + R + \ell + 2)}{2(-\Delta + \ell + 2)} (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(0)})^* \right), \quad (\text{C25})$$

$$a_4 = \frac{4(\Delta - 1)^2(-\Delta + \ell + 2)(\Delta + \ell)}{\Delta^2(\ell + 1 - \Delta)(\Delta + \ell + 1)(\ell + 2 - R - \Delta)(\ell + 2 + R - \Delta)(\Delta - R + \ell)(\Delta + R + \ell)} \times \\ \left[-\frac{(\Delta - R + \ell)(R(\ell(\ell + 2) - \Delta(\Delta + r^2 - 2)) + (\ell + 2 - \Delta)((\Delta + \ell)^2 - \Delta r^2))}{8(\Delta - 1)(\ell + 2 - \Delta)(\Delta + \ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(0)} \right. \\ \left. + \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(2)} + \frac{r(\Delta(R + 2 - \Delta) + \ell(\ell + 2))}{2(\ell + 2 - \Delta)(\Delta + \ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} + \frac{r(\Delta - R + \ell)}{4(\Delta + \ell)} \lambda_{\Phi_1\Phi_2\mathcal{O}}^{(1)} \right] \times \\ \left[-\frac{(\Delta - R + \ell)(R(\ell(\ell + 2) - \Delta(\Delta + r^2 - 2)) + (\ell + 2 - \Delta)((\Delta + \ell)^2 - \Delta r^2))}{8(\Delta - 1)(\ell + 2 - \Delta)(\Delta + \ell)} (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(0)})^* \right. \\ \left. + (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(2)})^* - \frac{\tilde{r}(\Delta(R + 2 - \Delta) + \ell(\ell + 2))}{2(\ell + 2 - \Delta)(\Delta + \ell)} (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(1)})^* - \frac{\tilde{r}(\Delta - R + \ell)}{4(\Delta + \ell)} (\lambda_{\Phi_4^\dagger\Phi_3^\dagger\mathcal{O}}^{(3)})^* \right]. \quad (\text{C26})$$

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